C. L. Liu

SOLUTIONS TO PROBLEMS IN

INTRODUCTION TO COMBINATORIAL MATHEMATICS

Prepared by

Murray Edelberg

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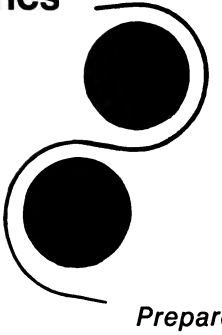

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Solutions to Problems in
INTRODUCTION TO COMBINATORIAL MATHEMATICS.

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Preface

I have been a teaching assistant for two offerings of a course in Combinatorial Mathematics, based on the textbook Introduction to Combinatorial Mathematics, by Professor C. L. Liu, which this solution book accompanies. It has been my experience, first in learning this material and then in helping to teach it, that solving a problem is often straightforward, once the right approach (or one of the right approaches) is taken. Discovering the right approach usually means trying several unfruitful approaches first, and this experience seems vital in developing skills for dealing with combinatorial problems.

This book is intended as an aid for instructors using the textbook. It contains solutions to every problem in the text, and may be helpful in selecting problems to assign, in developing new problems, and in preparing solutions for distribution to the class. Weekly problem assignments of five or six problems have been found appropriate.

For each problem, the statement of the problem is given, followed by its solution. A broken line separates the two. The symbol □ marks the end of each problem solution. All references to examples, lemmas, theorems and corollaries refer to the text.

I have tried to make each solution complete and clear. If I have succeeded it is largely due to the students with whom I have worked and their persistent desire to know. I would appreciate receiving any comments, suggestions or corrections that a reader of these solutions may care to make.

I would like to thank Professor Liu, who first taught me combinatorial mathematics and made it come alive with his great enthusiasm for the subject, for giving me the opportunity to prepare this solution book and for helping me with several of the problems, over which I might very well still be pondering without the benefit of his help.

Finally, I would like to thank Kathy Dimond, who zealously and meticulously arranged and typed this book, a somewhat tedious task. I believe this work gives evidence to her skill in technical typing.

Cambridge, Massachusetts
July, 1968

Murray Edelberg

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1-1. (a) Use the relation $C(n,r) = C(n-1,r) + C(n-1,r-1)$ to prove the identity

$$C(n+1,m) = C(n,m) + C(n-1,m-1) + C(n-2,m-2) + \cdots + C(n-m,0)$$

from $m \leq n$.

(b) Prove this identity using combinatorial arguments.

(a) $C(n+1,m) = C(n,m) + C(n,m-1)$

$C(n,m-1) = C(n-1,m-1) + C(n-1,m-2)$

$\vdots$

$C(n-(m-2),1) = C(n-(m-1),1) + C(n-(m-1),0)$

$C(n-(m-1),0) = C(n-m,0)$

Substituting upward,

$$C(n+1,m) = C(n,m) + C(n-1,m-1) + C(n-2,m-2) + \cdots + C(n-m,0) \quad m \leq n$$

(b) If m of the (n+1) objects are identified as special objects and denoted $s_1, s_2, \cdots, s_m$, then

$C(n+1,m) =$	$C(n,m)$ +	exclude s_1
	$C(n-1,m-1)$ +	include s_1, exclude s_2
	$C(n-2,m-2)$ +	include s_1, s_2; exclude s_3
	$\vdots$	
	$C(n-j,m-j)$ +	include $s_1, s_2, \cdots, s_j$; exclude s_{j+1}
	$\vdots$	
	$C(n-(m-1),1)$ +	include $s_1, s_2, \cdots, s_{m-1}$; exclude s_m
	$C(n-m,0)$	include $s_1, s_2, \cdots, s_m$.

The m special objects are used to subdivide the count into a number of mutually exclusive and collectively exhaustive parts.

□

1-2. (a) Prove the identity

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1$$

(b) Discuss the combinatorial significance of this identity.

(c) Show that any integer m can be expressed uniquely in the following form (factorial representation):

$$m = a_1 \cdot 1! + a_2 \cdot 2! + a_3 \cdot 3! + \cdots + a_i \cdot i! + \cdots$$

where $0 \leq a_i \leq i$ for $i = 1,2,\cdots$

(a) $(n+1)! = (n+1)\cdot n! = n\cdot n! + n!$

$n! = n\cdot(n-1)! = (n-1)\cdot(n-1)! + (n-1)!$

$(n-1)! = (n-1)\cdot(n-2)! = (n-2)\cdot(n-2)! + (n-2)!$

$\vdots$

$2! = 2\cdot 1! = 1\cdot 1! + 0!$

$0! = 1$

Substituting upward,

$$(n+1)! = n\cdot n! + (n-1)\cdot(n-1)! + (n-2)\cdot(n-2)! + \cdots + 1\cdot 1! + 1$$

(b) Denote (n+1) distinct objects $0_1, 0_2, \cdots, 0_{n+1}$
and (n+1) distinct positions $p_1, p_2, \cdots, p_{n+1}$.

$(n+1)! =$

$n\cdot n!$ +	0_1 not in p_1
$(n-1)\cdot(n-1)!$ +	0_1 in p_1, 0_2 not in p_2
$(n-2)\cdot(n-2)!$ +	0_1 in p_1, 0_2 in p_2, 0_3 not in p_3
$\vdots$	
$(n-j)\cdot(n-j)!$ +	0_1 in p_1, 0_2 in p_2, $\cdots$, 0_j in p_j, 0_{j+1} not in p_{j+1}
$\vdots$	
$1\cdot 1!$ +	0_1 in p_1, 0_2 in p_2, $\cdots$, 0_{n-1} in p_{n-1}, 0_n not in p_n
1	0_1 in p_1, 0_2 in p_2, $\cdots$, 0_{n-1} in p_{n-1}, 0_n in p_n

(c) Any integer n can be represented in this form.

$n = k_1\cdot 2 + a_1 \qquad 0 \le a_1 \le 1$

$k_1 = k_2\cdot 3 + a_2 \qquad 0 \le a_2 \le 2$

$k_2 = k_3\cdot 4 + a_3 \qquad 0 \le a_3 \le 3$

$\vdots$

The sequence $k_1, k_2, k_3, \cdots$ is strictly decreasing and terminates when some $k_j = 0$. Then, substituting upward,

$$n = a_1\cdot 1! + a_2\cdot 2! + a_3\cdot 3! + \cdots + a_j\cdot j!$$

The factorial representation is unique.

Suppose some integer n has two different representations,

$$n = a_1\cdot 1! + a_2\cdot 2! + a_3\cdot 3! + \cdots + a_i\cdot i! + \cdots$$

$$n = a'_1\cdot 1! + a'_2\cdot 2! + a'_3\cdot 3! + \cdots + a'_i\cdot i! + \cdots$$

Then

$$(a_1-a_1')\cdot 1! + (a_2-a_2')\cdot 2! + (a_3-a_3')\cdot 3! + \cdots + (a_i-a_i')\cdot i! + \cdots = 0$$

where $-i \le (a_i-a_i') \le i$

But, using the result of part (a), this implies

$$\begin{aligned} a_1-a_1' &= 0 \\ a_2-a_2' &= 0 \\ a_3-a_3' &= 0 \\ &\vdots \\ a_i-a_i' &= 0 \\ &\vdots \end{aligned}$$

□

1-3. It is clear that

$$P(n,n) = P(n,n-1)$$

but

$$P(n,n) \neq P(n,n-2)$$

Give a combinatorial explanation of these two relations.

Designate one of n distinct objects as a special object. There are P(n,n-1) ways to place the remaining (n-1) distinct objects into n distinct cells. For each, there is one way to place the special object in the one vacant cell, hence

$$P(n,n) = P(n,n-1)$$

For each of the P(n,n-2) ways of placing (n-2) distinct objects into n distinct cells, there are two ways of placing the remaining two objects in the two vacant cells, thus

$$P(n,n) = 2P(n,n-2)$$

□

1-4. Use a combinatorial argument to prove the identity

$$C(n,0) + C(n,1) + C(n,2) + \cdots + C(n,n) = 2^n$$

Using the rule of sum, the number of ways of selecting any number (including none) of a set of n distinct objects is

$$\sum_{r=0}^{n} C(n,r)$$

Equivalently, since each object must either be selected or not selected, the rule of product gives 2^n for this quantity.

□

1-5. (a) Show that

$$n \cdot C(n-1,r) = (r+1) \cdot C(n,r+1)$$

What is the combinatorial significance of this identity?

(b) Prove the identity

$$C(n,1) + 2 \cdot C(n,2) + 3 \cdot C(n,3) + \cdots + n \cdot C(n,n) = n \cdot 2^{n-1}$$

(a) $$n \cdot C(n-1,r) = n \cdot \frac{(n-1)!}{r!((n-1)-r)!} = (r+1) \cdot \frac{n!}{(r+1)!(n-(r+1))!} = (r+1) \cdot C(n,r+1)$$

From a set of n distinct objects, select two subsets, one containing a single object, the other r objects. One way to do this is to first select the single object and then the r objects. By the rule of product, there are

$$n \cdot C(n-1,r)$$

ways to do this. Alternatively, select (r+1) of the n objects and from these select one. There are

$$C(n,r+1) \cdot (r+1)$$

ways to do this. The selection processes described are equivalent, establishing the identity.

(b) $$\sum_{k=1}^{n} k \cdot C(n,k) = \sum_{k=0}^{n-1} (k+1) \cdot C(n,k+1)$$

$$= \sum_{k=0}^{n-1} n \cdot C(n-1,k) \qquad \text{(from part (a))}$$

$$= n \sum_{k=0}^{n-1} C(n-1,k)$$

$$= n \cdot 2^{n-1} \qquad \text{(using the identity of Problem 1-4)}$$

□

1-6. For a given n, show that C(n,k) is maximum when

$$k = \frac{n-1}{2}, \frac{n+1}{2} \quad \text{if n is odd}$$

$$k = \frac{n}{2} \quad \text{if n is even}$$

Consider the ratio

$$R(k) = \frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{n!}{(k+1)!(n-k-1)!} \cdot \frac{k!(n-k)!}{n!} = \frac{n-k}{k+1}$$

$R(k)$ is a monotonically decreasing function of k, for $k = 0,1,\cdots,n-1$, with $R(0) = n$ and $R(n-1) = \frac{1}{n}$.

For n-odd, $R(\frac{n-1}{2}) = 1$

so that $\binom{n}{k}$ is maximum for $k = \frac{n-1}{2}, \frac{n+1}{2}$.

For n-even, $R(\frac{n}{2}-1) = \frac{n+2}{n}$

and $R(\frac{n}{2}) = \frac{n}{n+2}$

So that $\binom{n}{k}$ is maximum for $k = \frac{n}{2}$.

□

1-7. (a) Use a combinatorial argument to prove that $(2n)!/2^n$ and $(3n)!/(2^n \cdot 3^n)$ are integers.

(b) Prove that $(n^2)!/(n!)^{n+1}$ is an integer.

(a) $\frac{(2n)!}{2^n}$ Consider 2n objects of n types, with 2 of each type. This is the number of distinct permutations possible.

$\frac{(3n)!}{2^n\ 3^n}$ Consider 3n objects of n types, with 3 of each type. This is the number of distinct permutations possible.

(b) $\frac{(n^2)!}{(n!)^{n+1}}$ Consider n^2 objects of n types, with n of each type. There are

$$\frac{(n^2)!}{(n!)^n}$$

distinct permutations of these objects. Partition this set of permutations into blocks, where two permutations are in the same block iff they differ only by a renaming of types.

For example, with n=2, the permutations 1221 and 2112 would belong to the same block.

There are n! permutations in each block, and

$$\frac{(n^2)!}{(n!)^{n+1}}$$

blocks, which therefore must be an integer.

1-8. Three integers are selected from the integers $1,2,\cdots,1,000$. In how may ways can these integers be selected such that their sum is divisible by 3?

Using the method of Example 1-12, the required result is

$$2\binom{333}{3} + \binom{334}{3} + (333)^2(334)$$

□

1-9. (a) Among $2n$ objects, n of them are identical. Find the number of ways to select n objects out of these $2n$ objects.

(b) Among $3n + 1$ objects, n of them are identical. Find the number of ways to select n objects out of these $3n + 1$ objects.

(a) Select any number of objects from the n distinct objects. For any selection of k of the n distinct objects, $k = 0,1,\cdots,n$, there is one way to add $(n-k)$ of the n identical objects. So, there are

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \text{ ways.}$$

(b) By a similar argument, the number of ways is

$$\sum_{k=0}^{n} \binom{2n+1}{k} = \frac{1}{2} \sum_{k=0}^{2n+1} \binom{2n+1}{k} = \frac{1}{2} \cdot 2^{2n+1} = 2^{2n}$$

□

1-10. From n distinct integers, two groups of integers are to be selected with k_1 integers in the first group and k_2 integers in the second group, where k_1 and k_2 are fixed and $k_1 + k_2 \leq n$. In how many ways can the selection be made such that the smallest integer in the first group is larger than the largest integer in the second group?

For any selection of $k_1 + k_2$ integers from n distinct integers, there is one way to partition the selection into two subsets containing k_1 and k_2 integers respectively, such that the smallest integer in the first subset is larger than the largest integer in the second subset. Therefore, there are

$$\binom{n}{k_1 + k_2}$$

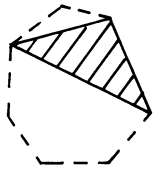
ways.

□

1-11. Suppose that no three of the diagonals of a convex n-gon meet at the same point inside of the n-gon. Find the number of different triangles the sides of which are made up of the sides of the n-gon, the diagonals, and segments of the diagonals.

Partition the set of triangles into four blocks, each characterized by the number of n-gon vertices and the number of diagonal intersections that form triangle vertices.

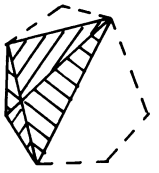
(i) 3 n-gon vertices - 0 diagonal intersections



Every 3 n-gon vertices specify one such triangle, giving

$\binom{n}{3}$ triangles of this type.

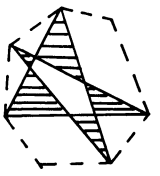
(ii) 2 n-gon vertices - 1 diagonal intersection



Every 4 n-gon vertices specify 4 such triangles, giving

$4\binom{n}{4}$ triangles of this type.

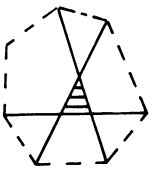
(iii) 1 n-gon vertex - 2 diagonal intersections



Every 5 n-gon vertices specify 5 such triangles, giving

$5\binom{n}{5}$ triangles of this type.

(iv) 0 n-gon vertices - 3 diagonal intersections



With the condition that no three diagonals meet at a point, every 6 n-gon vertices specify one such triangle, giving

$\binom{n}{6}$ triangles of this type.

Thus, the total number of triangles is

$$\binom{n}{3} + 4\binom{n}{4} + 5\binom{n}{5} + \binom{n}{6}$$

□

1-12. Consider the set of words of length n generated from the alphabet $\{0,1,2\}$.

(a) Show that the number of words in each of which the digit 0 appears an even number of times is $(3^n + 1)/2$.

(b) Prove the identity

$$\binom{n}{0}\cdot 2^n + \binom{n}{2}\cdot 2^{n-2} + \cdots + \binom{n}{q}\cdot 2^{n-q} = \frac{3^n + 1}{2}$$

where $q = n$ when n is even, and $q = n-1$ when n is odd.

(a) Partition the set of (3^n-1) words containing one or more 0's or 1's into blocks, where each block contains all words with a particular pattern of 2's. A block characterized by a given arrangement of k 2's contains 2^{n-k} members, half of which have an even number of 0's, by the result of Example 1-9. So, the number of words containing an even number of 0's (including the word of all 2's) is

$$\frac{3^n-1}{2} + 1 = \frac{3^n+1}{2}$$

(b) The number of words containing k 0's is

$$\binom{n}{k} \cdot 2^{n-k}$$

Summing over all even k, $0 \le k \le n$, and using the result of part (a), the given identity is obtained.

□

1-13. An alphabet of m letters can be transmitted through a communication channel. Find the number of different messages of n letters, if

(a) The letters can be used repeatedly in a message.

(b) ℓ of the m letters can be used only as the first and the last letters in a message; the other letters can appear anywhere with unrestricted repetitions in a message.

(c) ℓ of the m letters can be used only as the first and the last letters in a message; the other letters can appear anywhere, except the two ends, with unrestricted repetitions in a message.

(a) m^n

(b) $m^2(m-\ell)^{n-2}$

(c) $\ell^2(m-\ell)^{n-2}$

□

1-14. Five teaching machines are to be used by a group of m students. If the same number of students should be assigned to use the first and the second machines, in how many ways can the assignment be made?

$$\sum_{k=0}^{[\frac{m}{2}]} \binom{m}{k}\binom{m-k}{k} \cdot 3^{m-2k}$$

□

1-15. Among the set of 10^n n-digit integers, two integers are considered to be equivalent if one can be obtained by a permutation of the digits of the other.

(a) How many nonequivalent integers are there?

(b) If the digits 0 and 9 can appear at most once, how many nonequivalent integers are there, for $n \geq 2$?

(a) Select n of the 10 digits with repetitions allowed. There are

$$\binom{10+n-1}{n} = \binom{n+9}{n}$$

non-equivalent integers.

(b) Neither of the digits $\{0,9\}$ appears in

$$\binom{8+n-1}{n}$$ of these non-equivalent integers.

One of the digits $\{0,9\}$ appears in

$$2\binom{8+(n-1)-1}{n-1}$$ of these non-equivalent integers.

Both of the digits $\{0,9\}$ appear in

$$\binom{8+(n-2)-1}{n-2}$$ of these non-equivalent integers.

This gives a total of

$$\binom{n+7}{n} + 2\binom{n+6}{n-1} + \binom{n+5}{n-2}$$

non-equivalent integers in which the digits $\{0,9\}$ appear at most once.

□

1-16. In how many ways can the letters a,a,a,a,a,b,c,d,e be permuted such that no two a's are adjacent?

The four letters $\{b,c,d,e\}$ must be used as spacers between the five a's. There are 4! permutations in which no two a's are adjacent.

□

1-17. (a) A Boolean function of n variables is defined by the assignment of a value of either 0 or 1 to each of the 2^n n-digit binary numbers. How many different Boolean functions of n variables are there?

(b) A Boolean function can be represented conveniently in tabular form where all the n-digit binary numbers and their values are listed. Such a tabular form is called the truth table of a Boolean function. For example, the following table is the truth table of a Boolean function of three variables:

Three-digit binary number	Value
0 0 0	0
0 0 1	1
0 1 0	1
0 1 1	0
1 0 0	1
1 0 1	0
1 1 0	1
1 1 1	1

A self-dual Boolean function is one the truth table of which will remain unchanged after all the 0's and 1's in the table are interchanged. How many self-dual Boolean functions of n variables are there?

(c) A symmetric Boolean function is one the truth table of which will remain unchanged for any permutation of the n columns of the binary digits. How many symmetric Boolean functions of n variables are there?

(a) Since there are 2 ways to assign a value to each of the 2^n binary n-tuples, by the rule of product there are

$$\underbrace{2 \cdot 2 \cdot \cdots \cdot 2}_{2^n \text{ factors}} = 2^{2^n}$$

ways to assign all the values, and therefore 2^{2^n} different Boolean functions of n variables.

(b) Partition the set of 2^n binary n-tuples into 2^{n-1} blocks, each block containing an n-tuple and its 1's complement. In constructing a self-dual function, assigning a value to either member of a block fixes the value that must be given to the other member. So, independent value assignments may be made for only 2^{n-1} of the 2^n n-tuples. Thus, there are $2^{2^{n-1}}$ different self-dual Boolean functions of n variables.

(c) Partition the set of 2^n n-tuples into blocks, where two n-tuples are in the same block if they contain the same number of 1's. There are (n+1) such blocks. By permuting the binary digit columns, any n-tuple can be transformed into any other member of the same block. A symmetric function is one which assigns the same value to every member of a block. There are (n+1) independent value assignments to be made, and 2^{n+1} different symmetric Boolean functions.

□

1-18. (Statistical Mechanics: Bose-Einstein Counting) A system consists of four identical particles. The total energy of the system is equal to $4E_0$ where E_0 is a positive constant. Each of the particles can have an energy level equal to kE_0 (k = 0,1,2,3,4). A particle of energy kE_0 can occupy one of the k^2+1 distinct energy states at that energy level. How many different configurations, in terms of energy states occupied by the particles, can the system have?

First, assign 4 non-distinct quantities of energy (E_0) to 4 non-distinct particles. This is like the problem of partitioning the integer 4. For each such partition,

4
31
22
211
1111

there is one corresponding energy assignment, and conversely.

Second, for each of these energy level assignments, compute the number of possible system energy-state configurations.

Particles of the same energy level may occupy identical energy states. For particles sharing the same energy level, select energy states with repetition.

energy level (kE_0)	0	E_0	$2E_0$	$3E_0$	$4E_0$	
# of energy states (k^2+1)	1	2	5	10	17	
energy level assignments (# of particles)	3	-	-	-	1	$\binom{1+3-1}{3}\cdot\binom{17}{1} = 17$
	2	1	-	1	-	$\binom{1+2-1}{2}\cdot\binom{2}{1}\cdot\binom{10}{1} = 20$
	2	-	2	-	-	$\binom{1+2-1}{2}\cdot\binom{5+2-1}{2} = 15$
	1	2	1	-	-	$\binom{1}{1}\cdot\binom{2+2-1}{2}\cdot\binom{5}{1} = 15$
	-	4	-	-	-	$\binom{2+4-1}{4} = 5$
						Total 72

□

1-19. (Statistical Mechanics: Fermi-Dirac Counting) Consider a system that is the same as the one described in Prob. 1-18 except for the following:

1. At the energy level kE_0 there are $2(k^2+1)$ distinct energy states.
2. No two particles can be in the same energy state.

How many different configurations can the system have?

Particles of the same energy level must occupy different energy states. For particles sharing the same energy level, select energy states without repetition.

energy level (kE_0)	0	E_0	$2E_0$	$3E_0$	$4E_0$
# of energy states $2(k^2+1)$	2	4	10	20	34

energy level assignments (# of particles)					number of energy-state configurations
	3	-	-	-	1	cannot occur (since 3 > 2)
	2	1	-	1	-	$\binom{2}{2}\cdot\binom{4}{1}\cdot\binom{20}{1}$ = 80
	2	-	2	-	-	$\binom{2}{2}\cdot\binom{10}{2}$ = 45
	1	2	1	-	-	$\binom{2}{1}\cdot\binom{4}{2}\cdot\binom{10}{1}$ = 120
	-	4	-	-	-	$\binom{4}{4}$ = 1
						Total 246

□

1-20. (Statistical Machanics) A system consists of three identical particles, two of them in one potential well, the other in a separate potential well. The total energy of the system is equal to $3E_0$. Each of the particles can have an energy level equal to kE_0 (k = 0,1,2,3). A particle of energy kE_0 can occupy one of the $2(k^2+1)$ distinct energy states at that energy level. How many different configurations, in terms of the occupation of energy states in the two potential wells, can the system have?

---

energy level assignments to particles (in units of E_0)		numbers of energy state configurations
3	0 0	$\binom{20}{1}\cdot\binom{2+2-1}{2}$ = 60
2	0 1	$\binom{10}{1}\cdot\binom{2}{1}\cdot\binom{4}{1}$ = 80
1	0 2	$\binom{4}{1}\cdot\binom{2}{1}\cdot\binom{10}{1}$ = 80
1	1 1	$\binom{4}{1}\cdot\binom{4+2-1}{2}$ = 40
0	0 3	$\binom{2}{1}\cdot\binom{2}{1}\cdot\binom{20}{1}$ = 80
0	1 2	$\binom{2}{1}\cdot\binom{4}{1}\cdot\binom{10}{1}$ = 80
		total 420

□

1-21. A system has N molecules. Each molecule has an A-atom at a fixed location and two identical B-atoms which can occupy any two of four possible positions in the molecule.

(a) How many different configurations can the system have?

(b) If some molecules can have one or two extra B-atoms by taking them from other molecules, how many different configurations can the system have?

(c) Repeat part (b) if, instead of two identical B-atoms, each molecule contains a B-atom and a C-atom.

(a) There are $\binom{4}{2}$ configurations for each molecule and

$$\binom{4}{2}^N = 6^N$$

system configurations.

(b) Place 2N B-atoms in 4N distinct positions. There are

$$\binom{4N}{2N} \text{ system configurations.}$$

(c) There are 4^N ways to place the C-atoms. For each, there are $\binom{3N}{N}$ ways to place the B-atoms, giving

$$4^N \binom{3N}{N} \text{ system configurations.}$$

□

1-22. A system consists of N distinct particles each of which can have an energy level equal to kE_0 for $k = 0,1,2,\cdots$. If the total energy of the system is equal to ME_0, where M is a positive integer, in how many ways can the total energy be distributed among the particles?

Select M of the N distinct particles with repetition allowed. Assign energy kE_0 to the i^{th} particle, $i = 1,2,\cdots, N$, where k_i is the number of times the i^{th} particle is selected. There are

$$\binom{N+M-1}{M}$$

ways to distribute ME_0 units of energy among N distinct particles.

□

1-23. (Information Theory: Sphere Packing) A set of code words that are of the same length is called a block code. In binary block codes, the distance between two code words is defined as the number of digits where the two code words differ. For example, the distance between 0110 and 1011 is 3 and the distance between 0110 and 0111 is 1. When the minimum distance between any two code words in a set of n-digit code words is $2 \cdot r+1$, the set forms an r-error-correcting code of length n. Let A(n,r) denote the maximum number of code words in an r-error-correcting code of length n.

(a) (a) Show that

$$A(n,r) \leq \frac{2^n}{\sum_{j=0}^{r} C(n,j)} \quad \text{(Hamming bound)}$$

(b) Show that

$$A(n,r) \geq \frac{2^n}{\sum_{j=0}^{2r} C(n,j)} \quad \text{(Gilbert bound)}$$

Hint: Consider the following geometric interpretation. In an n-dimensional space in which the cartesian coordinates can assume only the two values 0 and 1, an n-digit binary word can be represented as a point. If a code is an r-error-correcting one, there is at most, one code word inside any "sphere" of diameter 2r.

(a) Associated with each code word is a "sphere" of radius r containing $\sum_{j=0}^{r} C(n,j)$ n-tuples. The central n-tuple is the code word and the remaining $\sum_{j=1}^{r} C(n,j)$ r-or-less distant n-tuples must all be decoded as that code word. A(n,r) is less than or equal to the number of such spheres (non-overlapping) that can be packed into this n-dimensional space, thus

$$A(n,r) \leq \frac{2^n}{\sum_{j=0}^{r} C(n,j)}$$

(b) Because the minimum distance between code words must be 2r+1, each code word requires that the $\sum_{j=0}^{2r} C(n,j)$ 2r-or-less distant n-tuples not be code words. For any selection of A(n,r) code words, there are at most A(n,r). $\sum_{j=0}^{2r} C(n,j)$ n-tuples that are eliminated.

If

$$A(n,r) \cdot \sum_{j=0}^{2r} C(n,j) < 2^n$$

then there is at least one n-tuple remaining which is (2r+1)-or-more distant from every code word, and thus can be used as an additional code word, contradicting the fact that A(n,r) is maximum. Therefore,

$$A(n,r) \geq \frac{2^n}{\sum_{j=0}^{2r} C(n,j)}$$

□

1-24. (Information Theory: Group Codes) A binary group code is a set of binary code words with the property that the modulo 2 sum of any two code words in the set is also a code word in the set. Let A(r) be the total number of code words in an r-error-correcting binary group code of length n. Show that

$$A(r) \leq \frac{2(2r+1)}{4r+2-n}$$

Hint: Show that the following are true:

1. Any column in the list of code words contains either half ones and half zeros or all zeros.

2. The list of code words contains at least $[A(r)-1]\cdot(2r+1)$ ones.

---

(1) Consider the j^{th} column in the list, $j = 1,2,\cdots,n$. Let $N_1(j)$ be the number of 1's in this column, and $N_0(j)$ the number of 0's.

Assume $N_1(j) > 0$.

Let W_1 be any code word whose j^{th} digit is 1. Taking the modulo 2 sum of W_1 and each of the $N_0(j)$ code words having 0 as as the j^{th} digit gives $N_0(j)$ distinct code words with 1 as the j^{th} digit. Therefore

$$N_1(j) \geq N_0(j)$$

Taking the modulo 2 sum of W_1 and the $N_1(j)$ code words with j^{th} digit 1 (including W_1) gives $N_1(j)$ distinct code words having j^{th} digit 0. Therefore

$$N_0(j) \geq N_1(j)$$

and so,

$$N_0(j) = N_1(j)$$

under the assumption, $N_1(j) > 0$.

(2) The modulo 2 sum of any code word with itself gives the all 0 word which must be a code word. Since the minimum distance between code words is (2r+1), each of the remaining (A(r)-1) code words contain at least (2r+1) 1's. So, there are at least

$$(A(r)-1)\cdot(2r+1)$$

1's in the list. But, from (1), there are at most

$$n \cdot \frac{A(r)}{2}$$

is in the list. So

$$n\,\frac{A(r)}{2} \geq (A(r)-1)(2r+1)$$

from which the given inequality follows.

□

1-25. How many permutations of the integers 1, 2, $\cdots$, n are there such that every integer is followed by (but not necessarily immediately followed by) an integer which differs from it by 1? For example, with n = 4, 1432 is an acceptable permutation but 2431 is not.

In an acceptable permutation, the rightmost k integers, k = 1,2,$\cdots$, n, must collectively form a set of k consecutive integers.

The leftmost integer can only be 1 or n. If it is 1, then the second integer can only be 2 or n. If it is n, then the second integer can only be 1 or (n-1).

Repeating this argument through the $(n-1)^{st}$ position reveals a sequence of (n-1) binary decisions to be made, giving a total of

$$2^{n-1}$$

acceptable permutations.

□

1-26. Let $a_1, a_2, \cdots, a_{2n}$ denote an ordered sequence of n 1's and n (-1)'s. Let f(k) be the sum of its first k digits. That is,

$$f(k) = \sum_{i=1}^{k} a_i$$

Find the number of such sequences that have the property

$$f(k) \geq 0 \quad \text{for } k = 1,2,\cdots, 2n$$

Hint: Define a new function g(k) such that

$$g(k) = \begin{cases} f(k) & 1 \leq k \leq m \\ -(f(k)+2) & m < k \leq 2n \end{cases}$$

where m is the smallest integer for which f(m) = -1, if there is such an integer. Otherwise m = 2n.

There are $\binom{2n}{n}$ distinct sequences $a_1, a_2, \cdots, a_{2n}$. For each, the sequence g(1), g(2), $\cdots$, g(2n) is unique, since it can be inverted as follows

$$a_1 = g(1)$$

$$a_k = \begin{cases} g(k)-g(k-1) & 2 \leq k \leq m \\ -(g(k)-g(k-1)) & m < k \leq 2n \end{cases}$$

where m is the smallest integer for which g(m) = -1, if there is such an integer; otherwise m = 2n.

For every acceptable sequence

$$g(2n) = f(2n) = 0.$$

However, for every unacceptable sequence

$$g(2n) = -(f(2n)+2) = -2$$

and $g(1), g(2), \cdots, g(2n)$ must contain $(n+1)$ unit "descents" and $(n-1)$ unit "ascents".

Thus there are $\binom{2n}{n+1}$ unacceptable sequences and

$$\binom{2n}{n} - \binom{2n}{n+1}$$

acceptable ones.

2-1. Evaluate the following sums in which n, m, and k are non-negative integers.

(a) $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{q}$ where q = n if n is even, and q = n-1 if n is odd.

(b) $\binom{n}{k} + \binom{n+1}{k} + \binom{n+2}{k} + \cdots + \binom{n+m}{k}$

(c) $\binom{2n}{0} - \binom{2n-1}{1} + \binom{2n-2}{2} - \cdots + (-1)^n \binom{n}{n}$

(d) $\binom{2n}{n} + 2\binom{2n-1}{n} + 2^2\binom{2n-2}{n} + \cdots + 2^n \binom{n}{n}$

(e) $\binom{n}{0}\binom{m}{k} + \binom{n}{1}\binom{m}{k-1} + \binom{n}{2}\binom{m}{k-2} + \cdots + \binom{n}{k}\binom{m}{0}$ $k \le \min(m,n)$

(f) $\binom{n}{0}\binom{n}{k} - \binom{n}{1}\binom{n-1}{k-1} + \binom{n}{2}\binom{n-2}{k-2} - \cdots + (-1)^k \binom{n}{k}\binom{n-k}{0}$

(g) $\binom{n}{k} + \binom{n-1}{k-1} + \binom{n-2}{k-2} + \cdots + \binom{n-k+1}{1} + \binom{n-k}{0}$

(h) $\binom{n}{0} + 2\binom{n}{1} + 2^2\binom{n}{2} + \cdots + 2^n\binom{n}{n}$

(a) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n$

$(1-x)^n = \binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 - \binom{n}{3}x^3 + \cdots + (-1)^n\binom{n}{n}x^n$

$$\frac{(1+x)^n + (1-x)^n}{2} = \binom{n}{0} + \binom{n}{2}x^2 + \cdots \binom{n}{q}x^q \quad \text{where } q = \begin{cases} n & n \text{ even} \\ n-1 & n \text{ odd} \end{cases}$$

$$\left.\frac{(1+x)^n + (1-x)^n}{2}\right|_{x=1} = \binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{q} = \begin{cases} 2^{n-1} & \text{for } n > 0 \\ 1 & \text{for } n = 0 \end{cases}$$

(b)
$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{k}x^k + \cdots + \binom{n}{n}x^n$$
$$(1+x)^{n+1} = \binom{n+1}{0} + \binom{n+1}{1}x + \cdots + \binom{n+1}{k}x^k + \cdots + \binom{n+1}{n}x^n + \binom{n+1}{n+1}x^{n+1}$$
$$(1+x)^{n+2} = \binom{n+2}{0} + \binom{n+2}{1}x + \cdots + \binom{n+2}{k}x^k + \cdots + \binom{n+2}{n}x^n + \binom{n+2}{n+1}x^{n+1} + \binom{n+2}{n+2}x^{n+2}$$
$$\vdots$$
$$(1+x)^{n+m} = \binom{n+m}{0} + \binom{n+m}{1}x + \cdots + \binom{n+m}{k}x^k + \cdots + \binom{n+m}{n}x^n + \binom{n+m}{n+1}x^{n+1} + \cdots + \binom{n+m}{n+m}x^{n+m}$$

It follows that the generating function A(x) for the desired sum (with k as the parameter) is

$$\begin{aligned} A(x) &= (1+x)^n + (1+x)^{n+1} + (1+x)^{n+2} + \cdots + (1+x)^{n+m} \\ &= \frac{(1+x)^n - (1+x)^{n+m+1}}{1 - (1+x)} \\ &= \frac{1}{x}\left[(1+x)^{n+m+1} - (1+x)^n\right] \end{aligned}$$

The coefficient of x^k in A(x) is

$$\binom{n+m+1}{k+1} - \binom{n}{k+1}$$

(c)

$$
\begin{array}{lll}
x^{2n}(1-x)^{2n} = & & \boxed{\binom{2n}{0}x^{2n} - \binom{2n}{1}x^{2n+1} + \cdots + \binom{2n}{2n}x^{4n}} \\
x^{2n-1}(1-x)^{2n-1} = & \binom{2n-1}{0}x^{2n-1} & \boxed{-\binom{2n-1}{1}x^{2n} + \binom{2n-1}{2}x^{2n+1} - \cdots - \binom{2n-1}{2n-1}x^{4n-2}} \\
x^{2n-2}(1-x)^{2n-2} = & \binom{2n-2}{0}x^{2n-2} - \binom{2n-2}{1}x^{2n-1} & \boxed{+\binom{2n-2}{2}x^{2n} - \binom{2n-2}{3}x^{2n+1} + \cdots + \binom{2n-2}{2n-2}x^{4n-4}} \\
\vdots & & \\
x^{n}(1-x)^{n} = & \binom{n}{0}x^{n} - \binom{n}{1}x^{n+1} + \cdots + (-1)^{n-2}\binom{n}{n-2}x^{2n-2} + (-1)^{n-1}\binom{n}{n-1}x^{2n-1} & \boxed{+(-1)^{n}\binom{n}{n}x^{2n}}
\end{array}
$$

The desired sum is the coefficient of x^{2n} in A(x) below,

$$A(x) = x^{2n}(1-x)^{2n} + x^{2n-1}(1-x)^{2n-1} + x^{2n-2}(1-x)^{2n-2} + \cdots + x^{n}(1-x)^{n}$$

In B(x) below, lower power terms have been added to simplify the following expressions. This addition does not alter the coefficient of x^{2n}.

$$B(x) = A(x) + x^{n-1}(1-x)^{n-1} + x^{n-2}(1-x)^{n-2} + \cdots + x^{0}(1-x)^{0}$$

$$= \frac{1-[x(1-x)]^{2n+1}}{1-x(1-x)} = \frac{1-[x(1-x)]^{2n+1}}{1-x+x^{2}} \cdot \frac{1+x}{1+x} = \{1-[x(1-x)]^{2n+1}\} \cdot \frac{1+x}{1+x^{3}}$$

$$= \{1-[x(1-x)]^{2n+1}\} \cdot (1+x)(1-x^{3}+x^{6}-x^{9}+\cdots+(-1)^{k}x^{3k}+\cdots)$$

The coefficient of x^{2n} in B(x) (and A(x)) is

$$\begin{cases} 1 & \text{for } 2n \equiv 0 \pmod 3 \\ -1 & \text{for } 2n \equiv 1 \pmod 3 \\ 0 & \text{for } 2n \equiv 2 \pmod 3 \end{cases}$$

(d)
$$
\begin{aligned}
(1+x)^{2n} &= \binom{2n}{0} + \binom{2n}{1}x + \cdots + \boxed{\binom{2n}{n}x^{n} +} \cdots + \binom{2n}{2n}x^{2n} \\
2(1+x)^{2n-1} &= 2\binom{2n-1}{0} + 2\binom{2n-1}{1}x + \cdots + \boxed{2\binom{2n-1}{n}x^{n} +} \cdots + 2\binom{2n-1}{2n-1}x^{2n-1} \\
4(1+x)^{2n-2} &= 4\binom{2n-2}{0} + 4\binom{2n-2}{1}x + \cdots + \boxed{4\binom{2n-2}{n}x^{n} +} \cdots + 4\binom{2n-2}{2n-2}x^{2n-2} \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
2^{n}(1+x)^{n} &= 2^{n}\binom{n}{0} + 2^{n}\binom{n}{1}x + \cdots + \boxed{2^{n}\binom{n}{n}x^{n}}
\end{aligned}
$$

The desired sum is the coefficient of x^n in A(x) below.

$$
\begin{aligned}
A(x) &= (1+x)^{2n} + 2(1+x)^{2n-1} + 4(1+x)^{2n-2} + \cdots + 2^{n}(1+x)^{n} \\
&= 2^{2n}\left[\left(\frac{1+x}{2}\right)^{2n} \left(\frac{1+x}{2}\right)^{2n-1} + \left(\frac{1+x}{2}\right)^{2n-2} + \cdots + \left(\frac{1+x}{2}\right)^{n}\right]
\end{aligned}
$$

Again, adding lower power terms without altering the coefficient of x^n,

$$
\begin{aligned}
B(x) &= 2^{2n}\left[\left(\frac{1+x}{2}\right)^{2n} + \left(\frac{1+x}{2}\right)^{2n-1} + \left(\frac{1+x}{2}\right)^{2n-2} + \cdots + \left(\frac{1+x}{2}\right)^{n} + \right. \\
&\qquad \left. \left(\frac{1+x}{2}\right)^{n-1} + \left(\frac{1+x}{2}\right)^{n-2} + \cdots + \left(\frac{1+x}{2}\right)^{0}\right] \\
&= 2^{2n}\left[\frac{1-\left(\frac{1+x}{2}\right)^{2n+1}}{1-\frac{1+x}{2}}\right] \\
&= 2^{2n+1}\left[\frac{1-\left(\frac{1+x}{2}\right)^{2n+1}}{1-x}\right] \\
&= 2^{2n+1}\left[1 - \frac{1}{2^{2n+1}}\left(\binom{2n+1}{0} + \binom{2n+1}{1}x + \cdots + \binom{2n+1}{2n+1}x^{2n+1}\right)\right] \cdot \\
&\qquad (1 + x + \cdots + x^{n} + \cdots)
\end{aligned}
$$

The coefficient of x^n in B(x) (and in A(x)) is

$$
\begin{aligned}
&2^{2n+1}\left[1 - \frac{1}{2^{2n+1}}\left(\binom{2n+1}{0} + \binom{2n+1}{1} + \cdots + \binom{2n+1}{n}\right)\right] = \\
&2^{2n+1}\left[1 - \frac{1}{2^{2n+1}} \cdot \frac{2^{2n+1}}{2}\right] = \\
&2^{2n}
\end{aligned}
$$

(e)
$$
\begin{aligned}
A(x) &= (1+x)^{n} \longleftrightarrow \left(\binom{n}{0}, \binom{n}{1}, \cdots, \binom{n}{k}, \cdots, \binom{n}{n}\right) \\
B(x) &= (1+x)^{m} \longleftrightarrow \left(\binom{m}{0}, \binom{m}{1}, \cdots, \binom{m}{k}, \cdots, \binom{m}{m}\right) \\
A(x)\cdot B(x) &= (1+x)^{n+m} \longleftrightarrow \left(\binom{n}{0}\binom{m}{0}, \binom{n}{0}\binom{m}{1} + \binom{n}{1}\binom{m}{0}, \binom{n}{0}\binom{m}{2} + \binom{n}{1}\binom{m}{1} + \binom{n}{2}\binom{m}{0},\right. \\
&\qquad \left. \cdots, \binom{n}{0}\binom{m}{k} + \binom{n}{1}\binom{m}{k-1} + \binom{n}{2}\binom{m}{k-2} + \cdots + \binom{n}{k}\binom{m}{0}, \cdots\right)
\end{aligned}
$$

A(x)·B(x) is the generating function for the desired sum (with k as parameter).

The coefficient of x^k in $A(x)\cdot B(x)$ is

$$\binom{n+m}{k}$$

when $k \le \min(n,m)$.

(f)
$$\sum_{i=0}^{k} (-1)^i \binom{n}{i}\binom{n-i}{k-i} = \sum_{i=0}^{k} (-1)^i \frac{n!}{(n-i)!\,i!} \cdot \frac{(n-i)!}{(n-k)!(k-i)!}$$

$$= \sum_{i=0}^{k} (-1)^i \frac{n!}{(n-k)!(k-i)!i!}$$

$$= \sum_{i=0}^{k} (-1)^i \frac{n!}{(n-k)!\,k!} \cdot \frac{k!}{(k-i)!\,i!}$$

$$= \sum_{i=0}^{k} (-1)^i \binom{n}{k}\binom{k}{i}$$

$$= \binom{n}{k} \sum_{i=0}^{k} (-1)^i \binom{k}{i}$$

$$= \binom{n}{k} (1-x)^k \Big|_{x=1}$$

$$= \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k > 0 \end{cases} \qquad n \ge 0$$

(g)
$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{k}x^k + \cdots + \binom{n}{n}x^n$$

$$x(1+x)^{n-1} = \binom{n-1}{0}x + \binom{n-1}{1}x^2 + \cdots + \binom{n-1}{k-1}x^k + \cdots + \binom{n-1}{n-1}x^n$$

$$x^2(1+x)^{n-2} = \binom{n-2}{0}x^2 + \cdots + \binom{n-2}{k-2}x^k + \cdots + \binom{n-2}{n-2}x^n$$

$$\vdots$$

$$x^k(1+x)^{n-k} = \binom{n-k}{0}x^k + \cdots + \binom{n-k}{n-k}x^n$$

The generating function $A(x)$ for the desired sum (with k as the parameter) is

$$A(x) = (1+x)^n + x(1+x)^{n-1} + x^2(1+x)^{n-2} + \cdots + x^k(1+x)^{n-k}$$

$$= x^n \left[\left(\frac{1+x}{x}\right)^n + \left(\frac{1+x}{x}\right)^{n-1} + \left(\frac{1+x}{x}\right)^{n-2} + \cdots + \left(\frac{1+x}{x}\right)^{n-k} \right]$$

Adding lower power terms without altering the coefficient of x^k,

$$B(x) = x^n \left[\left(\frac{1+x}{x}\right)^n + \left(\frac{1+x}{x}\right)^{n-1} + \left(\frac{1+x}{x}\right)^{n-2} + \cdots + \left(\frac{1+x}{x}\right)^{n-k} + \left(\frac{1+x}{x}\right)^{n-k-1} + \left(\frac{1+x}{x}\right)^{n-k-2} + \cdots + \left(\frac{1+x}{x}\right)^{0} \right]$$

$$= x^n \left[\frac{1 - \left(\frac{1+x}{x}\right)^{n+1}}{1 - \frac{1+x}{x}} \right]$$

$$= (1+x)^{n+1} - x^{n+1}$$

The coefficient of x^k in B(x) (and A(x)) is

$$\begin{cases} \binom{n+1}{k} & \text{for } 0 \le k \le n \\ 0 & \text{for } k = n+1 \end{cases}$$

(h)

$$\binom{n}{0} + \binom{n}{1} \cdot 2x + \binom{n}{2} \cdot 2^2x^2 + \cdots + \binom{n}{n} \cdot 2^n x^n = (1+2x)^n$$

$$\binom{n}{0} + \binom{n}{1} \cdot 2 + \binom{n}{2} \cdot 2^2 + \cdots + \binom{n}{n} \cdot 2^n = (1+2x)^n \Big|_{x=1} = 3^n$$

□

2-2 (a) Show that

$$\binom{m}{r+1} + \sum_{k=0}^{n-1} \binom{m+k}{r} = \binom{m+n}{r+1}$$

(b) Show that

$$\sum_{k=0}^{n-1} \binom{2+k}{2} = \binom{2+n}{3}$$

and thus

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{1}{3} n(n+1)(n+2)$$

(a) From Problem 2-1(b),

$$\sum_{k=0}^{n-1} \binom{m+k}{r} = \binom{m+n}{r+1} - \binom{m}{r+1}$$

(b)

$$\sum_{k=0}^{n-1} \binom{2+k}{2} = \binom{2+n}{3} - \cancel{\binom{2}{3}}$$

and,

$$\sum_{k=0}^{n-1} \binom{2+k}{2} = \frac{1}{2!} \sum_{k=0}^{n-1} (k+1)(k+2)$$

so,
$$\frac{1}{2!}\sum_{k=0}^{n-1}(k+1)(k+2) = \binom{n+2}{3} = \frac{1}{3!}\,n(n+1)(n+2)$$
$$\sum_{k=0}^{n-1}(k+1)(k+2) = \frac{1}{3}\,n(n+1)(n+2)$$

□

2-3. Among the three representatives from each of the 50 states, either none, or one, or two of them will be selected to form a special committee.

(a) In how many ways can the selection be made?

(b) If the committee has exactly 50 members, in how many ways can the selection be made? (The answer may be expressed as a summation.)

For a given state the enumerator is
$$\binom{3}{0} + \binom{3}{1}x + \binom{3}{2}x^2$$

The enumerator for the entire selection is
$$(1 + 3x + 3x^2)^{50}$$

a) $(1 + 3x + 3x^2)^{50}\Big|_{x=1} = 7^{50}$

b) Find the coefficient of x^{50} in
$$(1 + 3x + 3x^2)^{50} = [1 + 3x(1+x)]^{50}$$
$$= \sum_{r=0}^{50}\binom{50}{r}(3x)^r(1+x)^r$$

The coefficient of x^{50} is
$$\sum_{r=25}^{50}\binom{50}{r}\cdot 3^r\cdot\binom{r}{50-r}$$

□

2-4. Find the value of a_{50} in the following expansion:
$$\frac{x-3}{x^2-3x+2} = a_0 + a_1x + a_2x^2 + \cdots + a_{50}x^{50} + \cdots$$

$$\frac{x-3}{x^2-3x+2} = \frac{2}{x-1} - \frac{1}{x-2} = \frac{1}{2-x} - \frac{2}{1-x} = \frac{1}{2}\cdot\frac{1}{1-\frac{x}{2}} - \frac{2}{1-x}$$

$$a_{50} = \frac{1}{2}\cdot\frac{1}{2^{50}} - 2 = \frac{1}{2^{51}} - 2$$

□

2-5. In how many ways can 200 identical chairs be divided among four conference rooms such that each room will have 20 or 40 or 60 or 80 or 100 chairs?

The ordinary enumerator is

$$(x^{20} + x^{40} + x^{60} + x^{80} + x^{100})^4$$

and the desired result is the coefficient of x^{200}.

$$[x^{20}(1 + x^{20} + x^{40} + x^{60} + x^{80})]^4 = x^{80}\left(\frac{1-x^{100}}{1-x^{20}}\right)^4$$

$$(1-x^{100})^4 = 1 - 4x^{100} + 6x^{200} - 4x^{300} + x^{400}$$

$$(1-x^{20})^{-4} = 1 + \sum_{r=1}^{\infty} \frac{(-4)(-5)\cdots(-4-r+1)}{r!}(-x^{20})^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{4\cdot 5\cdot \cdots \cdot (r+3)}{r!} x^{20r}$$

$$= \sum_{r=0}^{\infty} \binom{r+3}{3} x^{20r}$$

So,

$$(x^{20} + x^{40} + x^{60} + x^{80} + x^{100})^4 = x^{80}(1 - 4x^{100} + 6x^{200} - 4x^{300} + x^{400}) \sum_{r=0}^{\infty} \binom{r+3}{3} x^{20r}$$

and the coefficient of x^{200} is

$$1 \cdot \binom{6+3}{3} - 4 \cdot \binom{1+3}{3} = 84 - 16 = 68$$

□

2-6. In how many ways can 3n letters be selected from 2n A's, 2n B's, and 2n C's?

Find the coefficient of x^{3n} in

$$A(x) = (1 + x + x^2 + \cdots + x^{2n})^3 = \left(\frac{1-x^{2n+1}}{1-x}\right)^3$$

$$(1-x)^{-3} = 1 + \sum_{r=1}^{\infty} \frac{(-3)(-4)\cdots(-r-2)}{r!}(-x)^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{3\cdot 4\cdot \cdots \cdot (r+2)}{r!} x^r$$

$$= \sum_{r=0}^{\infty} \binom{r+2}{2} x^r$$

$$A(x) = (1-3x^{2n+1} + 3x^{4n+2} - x^{6n+3}) \cdot \sum_{r=0}^{\infty} \binom{r+2}{2} x^r$$

The coefficient of x^{3n} in A(x) is

$$\binom{3n+2}{2} - 3\binom{n+1}{2}$$

□

2-7. In how many ways can n letters be selected from an unlimited supply of A's, B's, and C's if each selection must include an even number of A's?

Find the coefficient of x^n in

$$\frac{1}{1-x^2} \cdot \frac{1}{1-x} \cdot \frac{1}{1-x} = \frac{1}{(1-x)^3} \cdot \frac{1}{1+x}$$

Using a partial fraction expansion,

$$\frac{1}{(1-x)^3(1+x)} = \frac{\frac{1}{2}}{(1-x)^3} + \frac{\frac{1}{4}}{(1-x)^2} + \frac{\frac{1}{8}}{1-x} + \frac{\frac{1}{8}}{1-x}$$

The coefficient of x^n is

$$\frac{1}{2}\binom{n+2}{2} + \frac{1}{4}(n+1) + \frac{1}{8} + \frac{1}{8}(-1)^n$$

□

2-8. There are p kinds of objects with four of each kind and q kinds of objects with two of each kind (q is an even number). In how many ways can they be divided into two equal piles such that there is an even number of each kind of object in both piles? (0 is taken as an even number). Evaluate your result for p = 4 and q = 2.

The ordinary enumerator for the formation of either pile is

$$(1 + x^2 + x^4)^p \, (1 + x^2)^q$$

Let i index the number of x^4 factors and j the number of x^2 factors from $(1 + x^2 + x^4)^p$. The coefficient of x^k, k even, may be expressed as

$$\sum_{i=0}^{p} \sum_{j=0}^{p-i} \binom{p}{i}\binom{p-i}{j}\binom{q}{\frac{k}{2}-2i-j}$$

$$= \sum_{i=0}^{p} \binom{p}{i} \sum_{j=0}^{p-i} \binom{p-i}{j}\binom{q}{\frac{k}{2}-2i-j}$$

$$= \sum_{i=0}^{p} \binom{p}{i}\binom{p+q-i}{\frac{k}{2}-2i}$$

The coefficient of x^{2p+q}, q even, is

$$\sum_{i=0}^{p} \binom{p}{i}\binom{p+q-i}{p+\frac{q}{2}-2i} = \sum_{i=0}^{p} \binom{p}{i}\binom{p+q-i}{\frac{q}{2}+i}$$

with p = 4, q = 2, this gives

$$\binom{4}{0}\binom{6}{1} + \binom{4}{1}\binom{5}{2} + \binom{4}{2}\binom{4}{3} = 70 \text{ ways.}$$

□

2-9. (a) Let a_r denote the number of ways in which the sum r will show when two distinct dice are rolled. Find the ordinary generating function of the sequence $(a_0, a_1, a_2, \cdots)$.

(b) Let a_r denote the number of ways in which the sum r can be obtained by rolling a die any number of times. Show that the ordinary generating function of the sequence $(a_0, a_1, a_2, \cdots)$ is $(1 - x - x^2 - x^3 - x^4 - x^5 - x^6)^{-1}$.

(a) $A(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^2$

(b) $$\sum_{k=0}^{\infty} (x + x^2 + x^3 + x^4 + x^5 + x^6)^k$$

$$= \frac{1}{1 - (x + x^2 + x^3 + x^4 + x^5 + x^6)}$$

$$= (1 - x - x^2 - x^3 - x^4 - x^5 - x^6)^{-1}$$

□

2-10. (a) Find the ordinary generating function of the sequence $(a_0, a_1, a_2, \cdots)$ where a_r is the number of ways of selecting r objects from a set of six distinct objects, where each object can be selected no more than thrice.

(b) From the generating function found in part (a), determine the number of outcomes when three indistinguishable dice are rolled.

(a) $(1 + x + x^2 + x^3)^6$

(b) Find the coefficient of x^3 in the ordinary enumerator of part (a). This coefficient is

$$\underbrace{\binom{6}{3}}_{1\cdot1\cdot1\cdot x\cdot x\cdot x} + \underbrace{\binom{6}{1}\binom{5}{1}}_{1\cdot1\cdot1\cdot1\cdot x\cdot x^2} + \underbrace{\binom{6}{1}}_{1\cdot1\cdot1\cdot1\cdot1\cdot x^3} = 56$$

This result can be obtained directly using Equation 1-2.

$$\binom{6+3-1}{3} = 56$$

□

2-11. Find the ordinary generating function of the sequence $(a_0,a_1,a_2,\cdots)$ where a_r is the number of partitions of the integer r into distinct primes.

$$\prod_{p_i \text{ prime}} (1 + x^{p_i}) = (1+x^2)(1+x^3)(1+x^5)(1+x^7)\cdots$$

□

2-12. (a) Find the ordinary generating function of the sequence $(a_0,a_1,a_2,\cdots)$ where a_r is the number of ways in which the sum r will show when two distinct dice are rolled with the first one showing even and the second one showing odd.

(b) Find the ordinary generating function of the sequence $(a_0,a_1,a_2,\cdots)$ where a_r is the number of ways in which the sum r will show when 10 distinct dice are rolled, with five of them showing even and the other five showing odd.

(a) $(x^2 + x^4 + x^6)(x + x^3 + x^5)$

(b) $(x^2 + x^4 + x^6)^5(x + x^3 + x^5)^5$

□

2-13. (a) Find the ordinary generating function of the sequence $(a_0,a_1,a_2,\cdots)$ where a_r is the number of ways in which r letters can be selected from the alphabet $\{0,1,2\}$ with unlimited repetitions except that the letter 0 must be selected an even number of times.

(b) From the generating function found in part (a) find an explicit expression for the number of ways of making such a selection of k letters.

(a) $(\frac{1}{1-x^2})(\frac{1}{1-x})^2$

(b) $$(1-x)^{-2} = 1 + \sum_{r=1}^{\infty} \frac{(-2)(-3)\cdots(-2-r+1)}{r!}(-x)^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{2\cdot 3\cdot \cdots (r+1)}{r!} x^r$$

$$= \sum_{r=0}^{\infty} (r+1)x^r$$

$$(1-x^2)^{-1} = \sum_{r=0}^{\infty} x^{2r}$$

The coefficient of x^k in

$$\sum_{r=0}^{\infty} x^{2r} \cdot \sum_{r=0}^{\infty} (r+1)x^r$$

is

$$\begin{cases} 1 + 3 + 5 + \cdots + (k+1) = \dfrac{(k+2)^2}{4} & \text{for } k \text{ even} \\ 2 + 4 + 6 + \cdots + (k+1) = \dfrac{(k+1)(k+3)}{4} & \text{for } k \text{ odd} \end{cases}$$

□

2-14. A "crooked die" is one whose six faces are marked with 1,2,3,4,5, and 7. Let a_r denote the number of ways in which the sum of the scores in r tosses of the "crooked die" is an even number. Show that the ordinary generating function of the sequence $(a_0,a_1,a_2,\cdots)$ is $1/2[1/(1-x)^2]\{[1/(1-x)^4] + [1/(1+x)^4]\}$.

$(\frac{1}{1-x})^2$ Enumerates the 2 even-digit occurrences, with no restriction

$(\frac{1}{1-x})^4$ Enumerates the 4 odd-digit occurrences, with no restriction

$\frac{1}{2}[(\frac{1}{1-x})^4 + (\frac{1}{1+x})^4]$ Enumerates the 4 odd-digit occurrences, with the restriction that there be an even number of them

$(\frac{1}{1-x})^2 \cdot \frac{1}{2}[(\frac{1}{1-x})^4 + (\frac{1}{1+x})^4]$ Enumerates the ways in which any number of even digits and an even number of odd digits occur.

□

2-15. Let A(x) denote the ordinary generating function of the sequence of numbers $(a_0,a_1,a_2,\cdots)$. Find the sequence whose ordinary generating function is A(x)(1-x). (1-x is therefore called the difference operator.)

$$A(x)(1-x) = (a_0 + a_1x + a_2x^2 + \cdots + a_rx^r + \cdots)(1-x)$$

$$= a_0 + (a_1-a_0)x + (a_2-a_1)x^2 + \cdots + (a_r-a_{r-1})x^r + \cdots$$

so that

$$A(x)(1-x) \longleftrightarrow \{a_0,\ a_1-a_0,\ a_2-a_1,\ \cdots,\ a_r-a_{r-1},\ \cdots\}$$

□

2-16. Let A(x) be the ordinary generating function of the sequence $(a_0,a_1,a_2,\cdots)$. Find the ordinary generating function of the sequence $(q_0,q_1,q_2,\cdots)$, where $q_n = \sum_{i=n+1}^{\infty} a_i$. (Assume that all the q's are finite.)

$$A(x) \longleftrightarrow (a_0, a_1, \cdots, a_n, \cdots)$$

$$\frac{A(1)}{1-x} \longleftrightarrow \left(\sum_{i=0}^{\infty} a_i, \sum_{i=0}^{\infty} a_i, \cdots, \sum_{i=0}^{\infty} a_i, \cdots\right)$$

$$\frac{A(x)}{1-x} \longleftrightarrow \left(a_0, a_0+a_1, \cdots, \sum_{i=0}^{n} a_i, \cdots\right)$$

$$\frac{A(1) - A(x)}{1-x} \longleftrightarrow \left(\sum_{i=1}^{\infty} a_i, \sum_{i=2}^{\infty} a_i, \cdots, \sum_{i=n+1}^{\infty} a_i, \cdots\right)$$

□

2-17. It is known that the ordinary generating function of the sequence $1,b,b^2,\cdots,b^r,\cdots)$ is $1/(1-bx)$. Find the sequence whose ordinary generating function is $b^k x^k/(1-bx)^{k+1}$.

$$\frac{1}{1-bx} = \sum_{n=0}^{\infty} b^n x^n$$

$$\frac{d}{dx}\frac{1}{1-bx} = \frac{b}{(1-bx)^2} = \sum_{n=0}^{\infty} nb^n x^{n-1}$$

$$\frac{d^2}{dx^2}\frac{1}{1-bx} = \frac{2b^2}{(1-bx)^3} = \sum_{n=0}^{\infty} n(n-1)b^n x^{n-2}$$

$$\frac{d^3}{dx^3}\frac{1}{1-bx} = \frac{3\cdot 2b^3}{(1-bx)^4} = \sum_{n=0}^{\infty} n(n-1)(n-2)b^n x^{n-3}$$

$$\vdots$$

$$\frac{d^k}{dx^k}\frac{1}{1-bx} = \frac{k!b^k}{(1-bx)^{k+1}} = \sum_{n=0}^{\infty} n(n-1)(n-2)\cdots(n-k+1)b^n x^{n-k}$$

$$\frac{b^k}{(1-bx)^{k+1}} = \sum_{n=0}^{\infty} \frac{n!}{(n-k)!k!} b^n x^{n-k}$$

$$\frac{b^k x^k}{(1-bx)^{k+1}} = \sum_{n=0}^{\infty} \binom{n}{k} b^n x^n$$

□

2-18. Find the exponential generating function of the sequence $(1, 1\cdot 4, 1\cdot 4\cdot 7, \cdots, 1\cdot 4\cdot 7\cdot \cdots \cdot (3r+1), \cdots)$.

$$(1-3x)^{-\frac{4}{3}} = 1 + \sum_{r=1}^{\infty} \frac{(-\frac{4}{3})(-\frac{7}{3}) \cdots (-\frac{4}{3} - r+1)}{r!} (-3x)^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(\frac{4}{3})(\frac{7}{3}) \cdots (\frac{3r+1}{3})}{r!} 3^r x^r$$

$$= \sum_{r=0}^{\infty} \frac{1 \cdot 4 \cdot 7 \cdot \cdots (3r+1)}{r!} x^r$$

¤

2-19. Let a_r denote the number of ways of permuting r of the 10 letters A, A, A, A, B, C, C, D, E, E. Find the exponential generating function of the sequence $(a_0, a_1, a_2, \cdots)$.

$$\underbrace{(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!})}_{\text{A's}}\underbrace{(1 + x)}_{\text{B}}\underbrace{(1 + x + \frac{x^2}{2!})}_{\text{C's}}\underbrace{(1 + x)}_{\text{D}}\underbrace{(1 + x + \frac{x^2}{2!})}_{\text{E}} =$$

$$(1 + x)^2(1 + x + \frac{x^2}{2!})^2(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!})$$

□

2-20. Find the number of n-digit words generated from the alphabet $\{0,1,2,3\}$ in each of which the number of 0's is even.

The exponential enumerator is

$$\underbrace{(\frac{e^x + e^{-x}}{2})}_{\substack{\text{even \# of 0's}}}\cdot\underbrace{e^x}_{\substack{\text{any \# of 1's}}}\cdot\underbrace{e^x}_{\substack{\text{any \# of 2's}}}\cdot\underbrace{e^x}_{\substack{\text{any \# of 3's}}} = \frac{1}{2}(e^{4x} + e^{2x})$$

$n!\cdot$coefficient of $x^n = n!\cdot\frac{1}{2}(\frac{4^n}{n!} + \frac{2^n}{n!})$

$$= \frac{1}{2}(4^n + 2^n)$$

□

2-21. Find the number of n-digit words generated from the alphabet $\{0,1,2,3,4\}$ in each of which the total number of 0's and 1's is even.

The exponential enumerator is

$$\underbrace{(\frac{e^x + e^{-x}}{2})}_{\substack{\text{even \# of 0's}}}\cdot\underbrace{(\frac{e^x + e^{-x}}{2})}_{\substack{\text{even \# of 1's}}}\cdot\underbrace{e^x}_{\substack{\text{any \# of 2's}}}\cdot\underbrace{e^x}_{\substack{\text{any \# of 3's}}}\cdot\underbrace{e^x}_{\substack{\text{any \# of 4's}}} + \underbrace{(\frac{e^x - e^{-x}}{2})}_{\substack{\text{odd \# of 0's}}}\cdot\underbrace{(\frac{e^x - e^{-x}}{2})}_{\substack{\text{odd \# of 1's}}}\cdot\underbrace{e^x}_{\substack{\text{any \# of 2's}}}\cdot\underbrace{e^x}_{\substack{\text{any \# of 3's}}}\cdot\underbrace{e^x}_{\substack{\text{any \# of 4's}}}$$

$$= \frac{e^{3x}}{4}(e^{2x} + e^{-2x} + 2) + \frac{e^{3x}}{4}(e^{2x} + e^{-2x} - 2) = \frac{e^{3x}}{2}(e^{2x} + e^{-2x})$$

$$= \frac{1}{2}(e^{5x} + e^x)$$

$n!\cdot$coefficient of $x^n = n!\cdot\frac{1}{2}(\frac{5^n}{n!} + \frac{1}{n!}) = \frac{1}{2}(5^n+1)$

□

2-22. Find the number of n-digit words generated from the alphabet $\{0,1,2\}$ in each of which none of the digits appears exactly three times.

The exponential enumerator is

$$(e^x - \frac{x^3}{3!})^3 = e^{3x} - 3e^{2x}\frac{x^3}{3!} + 3e^x\frac{x^6}{3!3!} - \frac{x^9}{3!3!3!}$$

n! coefficient of x^n =

$$\begin{cases} 3^n & n = 0,1,2 \\ 3^n - 3\binom{n}{3}2^{n-3} & n = 3,4,5 \\ 3^n - 3\binom{n}{3}2^{n-3} + 3\frac{n!}{(n-6)!}\cdot\frac{1}{3!3!} & n \geq 6,\ n \neq 9 \\ 3^n - 3\binom{n}{3}2^{n-3} + 3\frac{n!}{(n-6)!}\cdot\frac{1}{3!3!} - \frac{n!}{3!3!3!} & n = 9 \end{cases}$$

Using the principle of inclusion and exclusion (Chapter Four), this is just $N(a_0'a_1'a_2')$ where a_i, $i = 0,1,2$, is defined as the property that a word contains exactly 3 of the digits i.

□

2-23. (a) Evaluate the definite integral $\int_0^\infty e^{-s}s^k\,ds$.

(b) Let A(x) and E(x) be the ordinary and exponential generating functions of the sequence $(a_0,a_1,a_2,\cdots)$, respectively. Show that

$$A(x) = \int_0^\infty e^{-s}E(sx)\,ds$$

(a) $\int_0^\infty e^{-s}s^k\,ds = -e^{-s}s^k\Big|_0^\infty + k\int_0^\infty e^{-s}s^{k-1}\,ds$

If $I_k = \int_0^\infty e^{-s}s^k\,ds$

then $I_k = k\,I_{k-1}$

and since $I_0 = \int_0^\infty e^{-s}\,ds = 1$

$$I_k = k!$$

(b) $E(x) = \sum_{k=0}^\infty a_k\frac{x^k}{k!}$

$$\int_0^\infty e^{-s}E(sx)\,ds = \int_0^\infty e^{-s}\left[\sum_{k=0}^\infty a_k\frac{(sx)^k}{k!}\right]ds$$

$$= \sum_{k=0}^\infty \frac{a_kx^k}{k!}\int_0^\infty e^{-s}s^k\,ds$$

$$= \sum_{k=0}^\infty a_kx^k = A(x)$$

□

2-24. (a) Let $A(x)$ be the ordinary generating function of the sequence of numbers $(a_0, a_1, a_2, \cdots)$. Find the sequences whose ordinary generating functions are

1. $\frac{d^j}{dx^j} A(x)$ where j is a non-negative integer

2. $x\frac{d}{dx}\left(x\frac{d}{dx}\left(x\frac{d}{dx} A(x)\right)\right)$

(b) Let θ denote the operator $x(d/dx)$, and let $P(\theta) = p_0 + p_1\theta + p_2\theta^2 + \cdots + p_k\theta^k$, where $p_0, p_1, p_2, \cdots, p_k$ are real constants. What is the sequence whose ordinary generating function is $P(\theta)A(x)$?

(a) (1) $$\frac{d^j}{dx^j} A(x) = \sum_{n=j}^{\infty} n(n-1)(n-2)\cdots(n-j+1)\, a_n x^{n-j} = \sum_{n=j}^{\infty} \frac{n!}{(n-j)!}\, a_n x^{n-j}$$

$$= \sum_{n=0}^{\infty} \frac{(j+n)!}{n!}\, a_{j+n}\, x^n$$

$$\frac{d^j}{dx^j} A(x) \longleftrightarrow \left(j!\, a_j,\ \frac{(j+1)!}{1!}\, a_{j+1},\ \frac{(j+2)!}{2!}\, a_{j+2},\ \cdots,\ \frac{(j+n)!}{n!}\, a_{j+n},\ \cdots \right)$$

(2) $$x\frac{d}{dx}\left(x\frac{d}{dx}\left(x\frac{d}{dx} A(x)\right)\right) = \sum_{n=0}^{\infty} n^3 a_n x^n \longleftrightarrow (0,\ 1^3a_1,\ 2^3a_2,\ 3^3a_3,\ 4^3a_4,\ \cdots,\ n^3a_n,\ \cdots)$$

(b) $$P(\theta)\, A(x) = p_0 \sum_{n=0}^{\infty} a_n x^n + p_1 \sum_{n=0}^{\infty} n\, a_n x^n + p_2 \sum_{n=0}^{\infty} n^2 a_n x^n + \cdots + p_k \sum_{n=0}^{\infty} n^k a_n x^n$$

$$= \sum_{i=0}^{k} p_i \sum_{n=0}^{\infty} n^i a_n x^n = \sum_{n=0}^{\infty} \sum_{i=0}^{k} p_i\, n^i\, a_n x^n$$

$$= \sum_{n=0}^{\infty} (p_0 + p_1 n + p_2 n^2 + \cdots + p_k n^k)\, a_n x^n$$

$$P(\theta)A(x) \longleftrightarrow (p_0a_0,\ (p_0 + p_1 + p_2 + \cdots + p_k)a_1,\ (p_0 + p_1 2 + p_2 2^2 + \cdots + p_k 2^k)a_2,\ \cdots$$
$$\cdots\ (p_0 + p_1 n + p_2 n^2 + \cdots + p_k \cdot n^k)a_n,\ \cdots)$$

□

2-25. Let $(p_0, p_1, p_2, \cdots)$ be a sequence of non-negative numbers which are less than or equal to 1. Let $P(x)$ denote the ordinary generating function of this sequence.

(a) The kth moment of the sequence $(p_0, p_1, p_2, \cdots)$, m_k, is defined as

$$m_k = \sum_{j=0}^{\infty} j^k p_j$$

Suppose that m_k exists for $k = 0, 1, 2, \cdots$. Show that the exponential generating function of the sequence $(m_0, m_1, m_2, \cdots)$, $M(x)$, is given by

$$M(x) = P(e^x)$$

[M(x) is also called the moment generating function.]

(b) The kth factorial moment of the sequence $(p_0,p_1,p_2, \cdots)$, n_k, is defined as

$$n_k = \sum_{j=k}^{\infty} \frac{j!}{(j-k)!} p_j$$

Show that the exponential generating function of the sequence $(n_0,n_1,n_2, \cdots)$, N(x), is given by P(x+1).

(a) $$M(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} p_j j^k\right) \frac{x^k}{k!} = \sum_{j=0}^{\infty} p_j \sum_{k=0}^{\infty} \frac{(jx)^k}{k!} = \sum_{j=0}^{\infty} p_j e^{jx}$$

$$= \sum_{j=0}^{\infty} p_j (e^x)^j = P(e^x)$$

(b) $$N(x) = \sum_{k=0}^{\infty} \left(\sum_{j=k}^{\infty} \frac{j!}{(j-k)!} p_j\right) \frac{x^k}{k!} = \sum_{j=0}^{\infty} p_j \sum_{k=0}^{j} \frac{j!}{(j-k)!k!} x^k$$

$$= \sum_{j=0}^{\infty} p_j (1+x)^j = P(x+1)$$

□

2-26. Let A(x) and B(x) be the ordinary generating functions of the two sequences of numbers $(a_0,a_1,a_2, \cdots)$ and $(b_0,b_1,b_2, \cdots)$, respectively.

(a) Define

$$c_{n,j} = b_0 c_{n-1,j} + b_1 c_{n-1,j-1} + \cdots + b_{j-1} c_{n-1,1} + b_j c_{n-1,0}$$

and

$$c_{1,j} = b_j$$

Suppose that $c_{n,j}$ exists for j = 0, 1, 2, $\cdots$. Find the ordinary generating function $C_n(x)$ of the sequence $(c_{n,0},c_{n,1},c_{n,2}, \cdots)$.

(b) Let $d_j = \sum_{k=0}^{\infty} a_k c_{k,j}$. Suppose that d_j exists for j = 0, 1, 2, $\cdots$. Show that the ordinary generating function of the sequence $(d_0,d_1,d_2, \cdots)$, D(x), is equal to A(B(x)).

(a) $$C_n(x) = \sum_{j=0}^{\infty} c_{n,j} x^j = \sum_{j=0}^{\infty} (b_0 c_{n-1,j} + b_1 c_{n-1,j-1} + \cdots + b_{j-1} c_{n-1,1} + b_j c_{n-1,0}) x^j$$

$$= B(x) C_{n-1}(x)$$

$$C_1(x) = \sum_{j=0}^{\infty} c_{1,j} x^j = \sum_{j=0}^{\infty} b_j x^j = B(x)$$

Therefore,

$$C_n(x) = B^n(x)$$

$$\text{(b)}\quad D(x) = \sum_{j=0}^{\infty} d_j x^j = \sum_{j=0}^{\infty} \Big(\sum_{k=0}^{\infty} a_k c_{k,j} \Big) x^j$$

$$= \sum_{k=0}^{\infty} a_k \sum_{j=0}^{\infty} c_{k,j} x^j = \sum_{k=0}^{\infty} a_k \, C_k(x) = \sum_{k=0}^{\infty} a_k \, B^k(x)$$

$$= A(B(x))$$

□

2-27. In this problem we investigate the following representation of an arbitrary positive integer n

$$n = \binom{b_1}{1} + \binom{b_2}{2} + \binom{b_3}{3} + \cdots + \binom{b_k}{k}$$

for a fixed k, where the b's satisfy the condition

$$0 \le b_1 < b_2 < b_3 < \cdots < b_k$$

That such a representation is possible can be seen from the following procedure for determining the b's:

b_k is the greatest integer such that $\binom{b_k}{k} \le n$

b_{k-1} is the greatest integer such that $\binom{b_{k-1}}{k-1} \le n - \binom{b_k}{k}$

.....

b_r is the greatest integer such that $\binom{b_r}{r} \le n - \sum_{j=r+1}^{k} \binom{b_j}{j}$

.....

b_1 is the greatest integer such that $\binom{b_1}{1} \le n - \sum_{j=2}^{k} \binom{b_j}{j}$

We can show that this representation is unique by proving the following inequality

$$\binom{n}{k} > \binom{n-1}{k} + \binom{n-2}{k-1} + \binom{n-3}{k-2} + \cdots + \binom{n-k}{1}$$

(a) Explain why.

(b) Show that

$$\binom{n}{k} - 1 = \binom{n-1}{k} + \binom{n-2}{k-1} + \binom{n-3}{k-2} + \cdots + \binom{n-k}{1}$$

(a) Suppose there are two different representations for some positive integer n.

$$n = \binom{b_k}{1} + \binom{b_2}{2} + \cdots + \binom{b_k}{k}$$

$$n = \binom{c_1}{1} + \binom{c_2}{2} + \cdots + \binom{c_k}{k}$$

Then

$$\binom{b_1}{1} + \binom{b_2}{2} + \cdots + \binom{b_k}{k} = \binom{c_1}{1} + \binom{c_2}{2} + \cdots + \binom{c_k}{k}$$

Let j be the largest index such that $b_j \neq c_j$.

Then

$$\binom{b_1}{1} + \binom{b_2}{2} + \cdots + \binom{b_j}{j} = \binom{c_1}{1} + \binom{c_2}{2} + \cdots + \binom{c_j}{j} \qquad [1]$$

Without loss of generality, assume $b_j > c_j$.

The proposed inequality gives

$$\binom{b_j}{j} > \binom{b_j-1}{j} + \binom{b_j-2}{j-1} + \cdots + \binom{b_j-j}{1}$$

which leads to

$$\binom{b_j}{j} > \binom{c_j}{j} + \binom{c_{j-1}}{j-1} + \cdots + \binom{c_1}{1},$$

a contradiction of [1].

Thus, $b_j = c_j$, $j = 1, 2, \cdots, k$.

(b) The right-hand side is the coefficient of x^k in

$$(1+x)^{n-1} + x(1+x)^{n-2} + x^2(1+x)^{n-3} + \cdots + x^{k-1}(1+x)^{n-k}$$

$$= (1+x)^{n-k}\left[(1+x)^{k-1} + x(1+x)^{k-2} + x^2(1+x)^{k-3} + \cdots + x^{k-1}\right]$$

$$= (1+x)^{n-k}\left[\frac{x^{k-1} - \frac{(1+x)^k}{x}}{1 - \frac{1+x}{x}}\right] = (1+x)^{n-k}\left[(1+x)^k - x^k\right]$$

$$= (1+x)^n - x^k(1+x)^{n-k}$$

The coefficient of x^k is equal to $\binom{n}{k} - \binom{n-k}{0} = \binom{n}{k} - 1$

□

2-28. Prove the identity

$$\frac{1}{1-x} = (1 + x + x^2 + \cdots + x^9)(1 + x^{10} + x^{20} + \cdots + x^{90})(1 + x^{100} + x^{200} + \cdots + x^{900}).$$

In terms of the partitions of integers, what is the significance of this identity?

$$\frac{1}{1-x} = \frac{1-x^{10}}{1-x} \cdot \frac{1-x^{100}}{1-x^{10}} \cdot \frac{1-x^{1000}}{1-x^{100}} \cdot \cdots \cdot \frac{1-x^{10^{k+1}}}{1-x^{10^k}} \cdot \cdots$$

$$= (1 + x + x^2 + \cdots + x^9)\cdot(1 + x^{10} + x^{20} + \cdots + x^{90})\cdot$$
$$(1 + x^{100} + x^{200} + \cdots + x^{900})\cdot(1 + x^{10^k} + x^{2\cdot 10^k} + \cdots + x^{9\cdot 10^k}) \cdots$$
$$= 1 + x + x^2 + x^3 + \cdots.$$

The partitioning of any integer into distinct parts of the form

$$k\cdot 10^n,\ k = 1, 2, \cdots, 9,\ n = 0, 1, 2, \cdots$$

can be done in exactly one way.

In other words, the decimal representation of an integer is unique.

□

2-29. Show that the number of partitions of the integer $2r + k$ into exactly $r + k$ parts is the same for any non-negative integer k.

Consider the three sets

S_1 all partitions of $(2r + k)$ into exactly $(r + k)$ parts

S_2 all partitions of $(2r + k)$ in which the largest part is $(r + k)$

S_3 all partitions of r

Taking the transpose of the Ferrers graph gives a one-to-one correspondence between S_1 and S_2.

$$|S_1| = |S_2|$$

Removing the part of size $(r + k)$ gives a one-to-one correspondence between S_2 and S_3.

$$|S_2| = |S_3|$$

So

$$|S_1| = |S_3|$$

But $|S_3|$ is independent of k.

□

2-30. Prove that the number of partititons of the integer n into m distinct parts is equal to the number of partitions of the integer $n-[m(m+1)/2]$ into at most m parts $(n > m(m+1)/2)$.

Hint: Use Ferrers graph.

In the Ferrers graph for a partition of n into m distinct parts, remove m dots from the first row, (m-1) dots from the second, (m-2) dots from the third, $\cdots$, 1 dot from the m^{th}. The result is a Ferrers graph for a partition of $[n - \frac{m(m+1)}{2}]$ into at most m parts.

The converse construction also holds.

This establishes the required 1:1 correspondence.

□

2-31. Show that the number of partitions of the integer 2n into three parts which are such that the sum of any two parts is greater than the third is equal to the number of partitions of n into exactly three parts.

For any partition of n into exactly 3 parts

$$a_1, a_2, a_3 > 0$$
$$a_1 + a_2 + a_3 = n$$

Let

$$A_1 = n-a_1$$
$$A_2 = n-a_2$$
$$A_3 = n-a_3$$

Then $A_1, A_2, A_3 > 0$

$$A_1 + A_2 + A_3 = 2n$$

and

$$A_1 + A_2 - A_3 = 2a_3 > 0$$
$$A_1 + A_3 - A_2 = 2a_2 > 0$$
$$A_2 + A_3 - A_1 = 2a_1 > 0$$

For any partition of 2n into three parts

$$A_1, A_2, A_3 > 0$$
$$A_1 + A_2 + A_3 = 2n$$

which satisfy the triangle inequality, let

$$2a_3 = A_1 + A_2 - A_3 > 0$$
$$2a_2 = A_1 + A_3 - A_2 > 0$$
$$2a_1 = A_2 + A_3 - A_1 > 0$$

Then

$$a_1 + a_2 + a_3 = \frac{1}{2}[A_1 + A_2 + A_3] = n$$

□

2-32. (The Theorems of Euler)

(a) A partition of an integer is said to be a self-conjugate partition if its corresponding Ferrers graph is symmetrical about its main diagonal; i.e., the transposition of the graph is identical to itself. Show that there is a one-to-one correspondence between the self-conjugate partitions and partitions into odd and unequal parts of an integer.

(b) Find the ordinary generating function of the sequence $(a_0, a_1, a_2, \cdots, a_r, \cdots)$ where a_r is the number of partitions of the integer r-N into even parts, the largest of which is less

than or equal to 2m. Clearly, a_r is the same as the number of partitions of the integer 1/2(r-N) such that the largest part in a partition is less than or equal to m.

(c) The largest square of dots at the upper left hand corner of a Ferrers graph (as illustrated in Fig. 2P-1) is called the Durfee square. Let a_r denote the number of

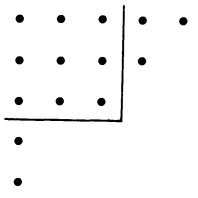


Figure 2P-1

self-conjugate partitions of the integer r whose Ferrers graphs contain an m × m Durfee square. Show that the ordinary generating function of the sequence $(a_0, a_1, a_2, \cdots, a_r, \cdots)$ is

$$\frac{x^{m^2}}{(1-x^2)(1-x^4)\cdots(1-x^{2m})}$$

(d) Use the results of parts (a), (b), and (c) to prove the identity

$$(1+x)(1+x^3)(1+x^5)\cdots = 1 + \frac{x}{1-x^2} + \frac{x^4}{(1-x^2)(1-x^4)} + \frac{x^9}{(1-x^2)(1-x^4)(1-x^6)} + \cdots = 1 + \sum_{m=1}^{\infty} \frac{x^{m^2}}{(1-x^2)(1-x^4)(1-x^6)\cdots(1-x^{2m})}$$

(First theorem of Euler)

(e) Prove the identity

$$(1+x^2)(1+x^4)(1+x^6)\cdots = 1 + \frac{x^2}{1-x^2} + \frac{x^6}{(1-x^2)(1-x^4)} + \frac{x^{12}}{(1-x^2)(1-x^4)(1-x^6)} + \cdots = 1 + \sum_{m=1}^{\infty} \frac{x^{m(m+1)}}{(1-x^2)(1-x^4)(1-x^6)\cdots(1-x^{2m})}$$

(Second theorem of Euler)

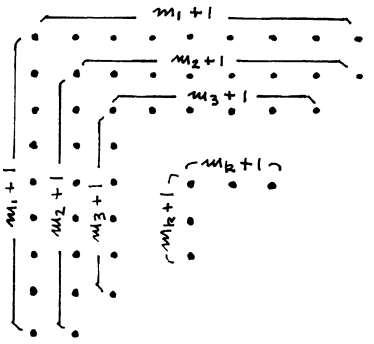
Hint: Replace x^2 by x. Instead of the Durfee squares in the Ferrers graphs, consider isosceles right triangles of dots at the upper left-hand corner of the Ferrers graphs.

(a) A partition of n into odd and unequal parts

$2m_1+1$ .....
$2m_2+1$ .....
$2m_3+1$ .....
⋮
$2m_k+1$ .....

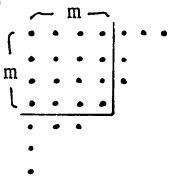
where $m_1 > m_2 > m_3 > \cdots > m_k$

may be transformed into the self-conjugate partition



and conversely.

(b) $$\frac{x^N}{(1-x^2)(1-x^4)\cdots(1-x^{2m})}$$

(c) 

Removal of the m×m Durfee square leaves $(n-m^2)$ dots which may be interpreted as two identical Ferrers graphs, each representing a partition of the integer $\frac{1}{2}(n-m^2)$ into parts $\leq m$. From (b), the number of self-conjugate partitions of n whose Ferrers graph contain an m×m Durfee square is enumerated by

$$\frac{x^{m^2}}{(1-x^2)(1-x^4)\cdots(1-x^{2m})}$$

(d) $(1+x)(1+x^3)(1+x^5)\cdots$ enumerates the partitions of n into odd and unequal parts. From (a), this also enumerates the self-conjugate partitions of n. Indexing the self-conjugate partitions by the dimension of their Durfee squares, and summing the result of (c) over all m, gives an alternate form for this enumerator, thereby establishing the identity.

(e) Replacing x^2 by x,

$$(1+x)(1+x^2)(1+x^3)\cdots = 1 + \sum_{m=1}^{\infty} \frac{x^{\frac{m(m+1)}{2}}}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)}$$

$(1+x)(1+x^2)(1+x^3)\cdots$ enumerates the partitions of n into distinct parts

$$\frac{x^{\frac{m(m+1)}{2}}}{(1-x)(1-x^2)(1-x^3)\cdots(1-x^m)}$$ enumerates

(i) the partitions of $[n-\frac{m(m+1)}{2}]$ into parts $\leq m$

(ii) the partitions of $[n-\frac{m(m+1)}{2}]$ into at most m parts

(iii) from Problem 2-30, and (ii) above, the partitions of n into m distinct parts.

Summing over all m, the result enumerates the partitions of n into distinct parts, establishing the identity.

□

2-33. Let a_r be the number of incongruent triangles with integral sides and perimeter r. Find the ordinary generating function of the sequence $(a_0, a_1, a_2, \cdots)$.

(i) n even

a_n = number of partitions of $\frac{n}{2}$ into exactly three parts (Problem 2-31)

= number of partitions of $\frac{n}{2}$ in which the largest part equals 3

= number of partitions of $\frac{n}{2} - 3$ into parts ≤ 3

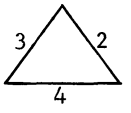
$$\frac{A(x) + A(-x)}{2} = \frac{x^6}{(1-x^2)(1-x^4)(1-x^6)} \qquad \text{(Problem 2-32(b))}$$

(ii) n odd

Every triangle of perimeter n becomes a triangle of perimeter n+3 when each edge length is increased by 1.

Every triangle of perimeter n+3 becomes a triangle of perimeter n when each edge length is decreased by 1.

The latter is not true for n even, e.g.,



$$\frac{2 \quad 1}{3}$$

To establish it for n odd, assume the contrary. Assume there is a triangle with integral side lengths a_1, a_2, a_3 for which

$$a_1 + a_2 + a_3 = n+3$$

$$a_1 + a_2 > a_3$$

$$a_1 + a_3 > a_2$$

$$a_2 + a_3 > a_1$$

and for which one or more of the following holds:

$$(a_1-1) + (a_2-1) \leq (a_3-1)$$

$$(a_1-1) + (a_3-1) \leq (a_2-1)$$

$$(a_2-1) + (a_3-1) \leq (a_1-1)$$

Without loss of generality, assume the first of these latter inequalities holds. Then

$$a_3 < a_1 + a_2 \leq a_3+1$$

and so,

$$a_1 + a_2 = a_3+1$$

But this, together with

$$a_1 + a_2 + a_3 = n+3$$

implies

$$a_3 = \frac{n}{2} + 1$$

which is not integral, and therefore a contradiction. So,

$$a_n = a_{n+3}$$

and, using the argument of part (i), (n+3 is even),

$$a_{n+3} = \text{number of partitions of } \frac{n+3}{2} - 3 \text{ into parts} \leq 3$$

$$\frac{A(x)-A(-x)}{2} = \frac{x^3}{(1-x^2)(1-x^4)(1-x^6)}$$

Finally,

$$A(x) = \frac{x^3 + x^6}{(1-x^2)(1-x^4)(1-x^6)}$$

□

3-1. (a) Find the ordinary generating functions of the sequences

1. $0, 0, 2\cdot1, 3\cdot2, \cdots, k(k-1)$
2. $0, 1^2, 2^2, 3^2, \cdots, k^2$
3. $0, 1\cdot3, 2\cdot4, 3\cdot5, \cdots, k(k+2)$
4. $0, 1\cdot2\cdot3, 2\cdot3\cdot4, \cdots, k(k+1)(k+2)$

(b) Evaluate the sums

1. $S_m = \sum_{k=0}^{m} k(k-1)$
2. $S_m = \sum_{k=0}^{m} k^2$
3. $S_m = \sum_{k=0}^{m} k(k+2)$
4. $S_m = \sum_{k=0}^{m} k(k+1)(k+2)$

(a) (1) $$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} kx^{k-1}$$

$$\frac{d}{dx}\left[\frac{1}{(1-x)^2}\right] = \frac{2}{(1-x)^3} = \sum_{k=0}^{\infty} k(k-1)x^{k-2}$$

$$\frac{2x^2}{(1-x)^3} = \sum_{k=0}^{\infty} k(k-1)x^k$$

(2) $$\frac{x}{(1-x)^2} = \sum_{k=0}^{\infty} kx^k$$

$$x\frac{d}{dx}\left[\frac{x}{(1-x)^2}\right] = \frac{x(1+x)}{(1-x)^3} = \sum_{k=0}^{\infty} k^2x^k$$

(3) $$\frac{x^3}{(1-x)^2} = \sum_{k=0}^{\infty} kx^{k+2}$$

$$\frac{1}{x}\frac{d}{dx}\left[\frac{x^3}{(1-x)^2}\right] = \frac{x(3-x)}{(1-x)^3} = \sum_{k=0}^{\infty} k(k+2)x^k$$

(4) $$\frac{2}{(1-x)^3} = \sum_{k=0}^{\infty} k(k-1)x^{k-2}$$

$$x\frac{d}{dx}\left[\frac{2}{(1-x)^3}\right] = \frac{6x}{(1-x)^4} = \sum_{k=0}^{\infty} k(k-1)(k-2)x^{k-2} = \sum_{k=0}^{\infty} k(k+1)(k+2)x^k$$

(b) Apply the summing operator to results of part (a).

(1) $$S(x) = \frac{2x^2}{(1-x)^4} = 2x^2 \sum_{r=0}^{\infty} \binom{r+3}{3} x^r$$

$$s_m = 2\binom{m-2+3}{3} = 2\binom{m+1}{3} = \sum_{k=0}^{m} k(k-1)x^k$$

(2) $$S(x) = \frac{x(1+x)}{(1-x)^4} = (x+x^2) \sum_{r=0}^{\infty} \binom{r+3}{3} x^r$$

$$s_m = \binom{m-1+3}{3} + \binom{m-2+3}{3} = \binom{m+2}{3} + \binom{m+1}{3} = \sum_{k=0}^{m} k^2$$

(3) $$S(x) = \frac{x(3-x)}{(1-x)^4} = (3x-x^2) \sum_{r=0}^{\infty} \binom{r+3}{3} x^r$$

$$s_m = 3\binom{m-1+3}{3} - \binom{m-2+3}{3} = 3\binom{m+2}{3} - \binom{m+1}{3} = \sum_{k=0}^{m} k(k+2)$$

(4) $$S(x) = \frac{6x}{(1-x)^5} = 6x \sum_{r=0}^{\infty} \binom{r+4}{4} x^r$$

$$s_m = 6\binom{m-1+4}{4} = 6\binom{m+3}{4} = \sum_{k=0}^{m} k(k+1)(k+2)$$

□

3-2. A circle and n straight lines are drawn on a plane. Each of these lines intersects all the other lines inside the circle. If no three or more lines meet at one point, into how many regions do these lines divide the circle?

Let a_n be the number of regions into which the circle is divided by n lines.

The n^{th} line intersects each of the other (n-1) lines exactly once within the circle, and is divided into n segments by these (n-1) intersections. Each such segment divides a previously existing region into two parts, giving the recurrence relation

$$a_n = a_{n-1} + n \qquad n \geq 1$$

with boundary condition $a_0 = 1$.

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} n x^n$$

$$A(x) - a_0 = xA(x) + \frac{x}{(1-x)^2}$$

$$A(x)(1-x) = 1 + \frac{x}{(1-x)^2}$$

$$A(x) = \frac{1}{1-x} + \frac{x}{(1-x)^3}$$

$$a_n = 1 + \frac{n}{2}(n+1)$$

□

3-3. A circle is drawn on a plane, and n straight lines are then drawn one by one. The first line passes through the circle. The rth line intersects only one of the other r-1 lines inside the circle if r is odd and greater than 1. The rth line intersects all the other r-1 lines inside the circle if r is even. If no three or more lines meet at one point inside the circle, into how many regions will n lines divide the circle, when n is even? when n is odd?

Let a_n be the number of regions into which the circle is divided by n lines.

n even $\quad a_n = a_{n-1} + n$ (as in Problem 3-2)

n odd $\quad a_n = a_{n-1} + 2$

Solve the seven and odd cases separately.

(i) n even

$$a_n = (a_{n-2} + 2) + n = a_{n-2} + n + 2 \qquad \text{for } n = 2, 4, 6, \cdots$$

$$\sum_{n=2,4,6,\cdots} a_n x^n = \sum_{n=2,4,6,\cdots} a_{n-2} x^n + \sum_{n=2,4,6,\cdots} n x^n + 2 \sum_{n=2,4,6,\cdots} x^n$$

$$\frac{A(x) + A(-x)}{2} - a_0 = x^2 \left[\frac{A(x) + A(-x)}{2}\right] + \left[\frac{\frac{x}{(1-x)^2} + \frac{-x}{(1+x)^2}}{2}\right] + \left[\frac{2}{1-x^2} - 2\right]$$

Choose $a_0 = 0$, so that $a_2 = a_0 + 2 + 2 = 4$

$$\left[\frac{A(x) + A(-x)}{2}\right](1-x^2) = \frac{2x^2}{(1-x^2)^2} + \frac{2}{1-x^2} - 2 = \frac{4x^2 - 2x^4}{(1-x^2)^2}$$

$$\left[\frac{A(x) + A(-x)}{2}\right] = \frac{4x^2 - 2x^4}{(1-x^2)^3}$$

$$= (4x^2 - 2x^4) \sum_{r=0}^{\infty} \binom{r+2}{2} x^{2r}$$

$$a_n = 4\binom{\frac{n}{2}-1+2}{2} - 2\binom{\frac{n}{2}-2+2}{2} = \frac{n}{4}(n+6) \qquad \text{for } n = 2,4,6,\cdots$$

(ii) n odd

$$a_n = (a_{n-2} + n-1) + 2 = \begin{cases} a_{n-2} + n + 1 & \text{for } n = 3,5,7,\cdots \\ 2 & \text{for } n = 1 \end{cases}$$

$$\sum_{n=3,5,7,\cdots} a_n x^n = \sum_{n=3,5,7,\cdots} a_{n-2} x^n + \sum_{n=3,5,7,\cdots} n x^n + \sum_{n=3,5,7,\cdots} x^n$$

$$\frac{A(x)-A(-x)}{2} - a_1 x = x^2\left[\frac{A(x)-A(-x)}{2}\right] + \left[\frac{\frac{x}{(1-x)^2} - \frac{-x}{(1-x)^2}}{2} - x\right] + \left[\frac{x}{1-x^2} - x\right]$$

$$\left[\frac{A(x)-A(-x)}{2}\right](1-x^2) = \frac{x+x^3}{(1-x^2)^2} + \frac{x}{1-x^2} = \frac{2x}{(1-x^2)^2}$$

$$\left[\frac{A(x)-A(-x)}{2}\right] = \frac{2x}{(1-x^2)^3} = 2x \sum_{r=0}^{\infty} \binom{r+2}{2} x^{2r}$$

$$a_n = 2\binom{\frac{n-1}{2}+2}{2} = \frac{(n+1)(n+3)}{4} \qquad \text{for } n = 1,3,5,7,\cdots$$

□

3-4. Let a_n be the number of incongruent triangles with integral sides and perimeter n.

(a) Show that

$$a_n = a_{n-3} \qquad \text{if n is even}$$

$$a_n = a_{n-3} + \frac{n + (-1)^{(n+1)/2}}{4} \qquad \text{if n is odd}$$

(b) Find the ordinary generating function of the sequence $(a_0, a_1, a_2, \cdots)$.

(a) n-even

As shown in the solution to Problem 2-33,

$$a_n = a_{n-3}$$

n-odd

Divide the triangles of perimeter n into two categories

(i) Those obtainable by adding 1 to each side of a triangle of perimeter n-3

(ii) Those not so obtainable.

There are a_{n-3} triangles in the first category.

Let b_1, b_2, b_3 be the sides of a triangle in the second category. That is

$$b_1 + b_2 + b_3 = n$$

$$b_1 + b_2 > b_3$$

$$b_1 + b_3 > b_2$$

$$b_2 + b_3 > b_1$$

and at least one of the following inequalities holds

$$(b_1-1) + (b_2-1) \leq (b_3-1)$$

$$(b_1-1) + (b_3-1) \leq (b_2-1)$$

$$(b_2-1) + (b_3-1) \leq (b_1-1)$$

Suppose the first of these holds. Then

$$b_3 < b_1 + b_2 \leq b_3 + 1$$

So

$$b_1 + b_2 = b_3+1$$

which, together with

$$b_1 + b_2 + b_3 = n$$

gives

$$b_1 + b_2 = \frac{n+1}{2}$$

$$b_3 = \frac{n-1}{2}$$

Each triangle of this type is characterized by a side of length $\frac{n-1}{2}$. The number of such triangles is equal to the number of partitions of $\frac{n+1}{2}$ into two parts.

If $\frac{n+1}{2}$ is even, there are $\frac{n+1}{4}$ such partitions.

If $\frac{n+1}{2}$ is odd, there are $\frac{n-1}{4}$ such partitions.

So, $a_n = a_{n-3} + \frac{n+(-1)^{\frac{n+1}{2}}}{4}$

(b) $\sum_{n = 4,6,8,\cdots} a_n x^n = \sum_{n = 4,6,8,\cdots} a_{n-3} x^n$

$$\frac{A(x)+A(-x)}{2} - a_0 - a_2x^2 = x^3 \left[\frac{A(x)-A(-x)}{2}\right]$$

$$a_0 = a_2 = 0$$

$$\frac{A(x)+A(-x)}{2} = x^3\left[\frac{A(x)-A(-x)}{2}\right] \qquad [1]$$

$$\sum a_n x^n = \sum a_{n-3} x^n + \frac{1}{4}\sum n x^n + \frac{1}{4}\sum (-1)^{\frac{n+1}{2}} x^n$$

$n = 3,5,7,\cdots \quad n = 3,5,7,\cdots \quad n = 3,5,7,\cdots \quad n = 3,5,7,\cdots$

$$\frac{A(x)-A(-x)}{2} - a_1 x = x^3\left[\frac{A(x)+A(-x)}{2}\right] + \frac{1}{4}\left[\frac{\frac{x}{(1-x)^2} - \frac{-x}{(1+x)^2}}{2} - x\right] + \frac{1}{4}\cdot\frac{x^3}{1+x^2}$$

$$a_1 = 0$$

$$\frac{A(x)-A(-x)}{2} = x^3\left[\frac{A(x)+A(-x)}{2}\right] + \frac{x^3}{(1-x^2)(1-x^4)} \qquad [2]$$

Combining [1] and [2],

$$\frac{A(x) + A(-x)}{2} = \frac{x^6}{(1-x^2)(1-x^4)(1-x^6)}$$

$$\frac{A(x) - A(-x)}{2} = \frac{x^3}{(1-x^2)(1-x^4)(1-x^6)}$$

$$A(x) = \frac{x^3 + x^6}{(1-x^2)(1-x^4)(1-x^6)}$$

□

3-5. (a) Show that the number of incongruent triangles whose sides are integral numbers with the longest one equal to ℓ is

$1/4(\ell+1)^2$ if ℓ is odd

$\frac{\ell}{4}(\ell+2)$ if ℓ is even

(b) Let f_n be the number of triangles every side of which does not exceed 2n, and let g_n be the number of triangles every side of which does not exceed 2n + 1. Find the expressions for f_n and g_n.

(a) The remaining two side lengths a_1 and a_2 must be chosen from

$$\{1,2,\cdots,\ell\}$$

subject to the condition

$$a_1 + a_2 > \ell$$

If $a_1 = k$, then a_2 can take any one of the k values

$$\ell,\ \ell-1,\ \cdots,\ \ell-k+1$$

Letting k range from 1 to ℓ gives a count of

$$\sum_1^{\ell} k = \frac{\ell}{2}(\ell+1) \text{ triangles.}$$

However, although each triangle for which $a_1 = a_2$ is counted once, each triangle for which $a_1 \neq a_2$ is counted twice.

Let E be the number of triangles for which $a_1 = a_2$.

Then the desired result is

$$\frac{1}{2}[\frac{\ell}{2}(\ell+1) + E]$$

and since

$$E = \begin{cases} \frac{\ell+1}{2} & \ell \text{ odd} \\ \frac{\ell}{2} & \ell \text{ even} \end{cases}$$

This becomes

$$\begin{cases} \frac{1}{4}(\ell+1)^2 & \ell \text{ odd} \\ \frac{\ell}{4}(\ell+2) & \ell \text{ even} \end{cases}$$

(b)

$$f_n - g_{n-1} = \frac{2n}{4}(2n+2) = n(n+1)$$

$$g_n - f_n = \frac{1}{4}[(2n+1)+1]^2 = (n+1)^2$$

$$\sum_{n=1}^{\infty} f_n x^n - \sum_{n=1}^{\infty} g_{n-1} x^n = \sum_{n=1}^{\infty} n(n+1)x^n$$

$$F(x) - f_0 - x\,G(x) = \frac{2x}{(1-x)^3}$$

$$\sum_{n=0}^{\infty} g_n x^n - \sum_{n=0}^{\infty} f_n x^n = \sum_{n=0}^{\infty} (n+1)^2 x^n$$

$$G(x) - F(x) = \frac{1+x}{(1-x)^3}$$

$$F(x) = x[F(x) + \frac{1+x}{(1-x)^3}] + \frac{2x}{(1-x)^3}$$

$$F(x) = \frac{x^2+3x}{(1-x)^4} = (x^2+3x) \sum_{r=0}^{\infty} \binom{r+3}{3} x^r$$

$$f_n = \binom{n-2+3}{3} + 3\binom{n-1+3}{3}$$

$$= \binom{n+1}{3} + 3\binom{n+2}{3}$$

$$G(x) = \frac{x^2+3x}{(1-x)^4} + \frac{1+x}{(1-x)^3} = \frac{3x+1}{(1-x)^4} = (3x+1)\sum_{r=0}^{\infty}\binom{r+3}{3}x^r$$

$$g_n = 3\binom{n-1+3}{3} + \binom{n+3}{3}$$

$$= 3\binom{n+2}{3} + \binom{n+3}{3}$$

□

3-6. In the linear resistance network shown in Fig. 3P-1, find the voltage v_0.
Hint: Assume first that $v_0 = 1$.

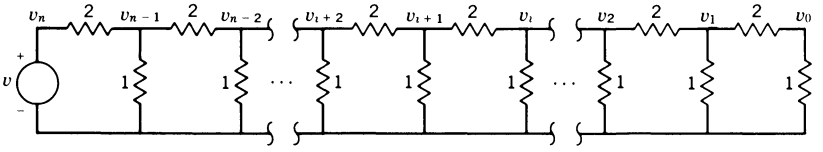


Figure 3P-1

Let $v_0 = 1$

$$\frac{v_{i+2} - v_{i+1}}{2} = v_{i+1} + \frac{v_{i+1} - v_i}{2}$$

$$v_{i+2} - 4v_{i+1} + v_i = 0 \qquad \text{for } i \geq 0$$

Assume $v_i = A\alpha^i$

Then $\alpha^2 - 4\alpha + 1 = 0$

$$\alpha = 2 \pm \sqrt{3}$$

$$v_i = A_1(2+\sqrt{3})^i + A_2(2-\sqrt{3})^i$$

Using the boundary conditions $v_0 = 1$ gives

$$v_1 = 3$$

$$A_1 = \frac{1+\sqrt{3}}{2\sqrt{3}}, \quad A_2 = -\frac{1-\sqrt{3}}{2\sqrt{3}}$$

and

$$v_n = \frac{1}{2\sqrt{3}}\left[(1+\sqrt{3})(2+\sqrt{3})^n - (1-\sqrt{3})(2-\sqrt{3})^n\right] = \frac{v_n}{v_0}$$

Finally, $v_n = V$ and

$$v_0 = \frac{2\sqrt{3}\,V}{\left[(1+\sqrt{3})(2+\sqrt{3})^n - (1-\sqrt{3})(2-\sqrt{3})^n\right]}$$

□

3-7. (a) How many n-digit binary sequences that have no adjacent 0's are there?

(b) How many n-digit ternary sequences that have no adjacent 0's are there?

(a) Let a_n = number of binary n-tuples having no adjacent 0's.

b_n = number of binary n-tuples having no adjacent 0's and ending in 1.

Then

$$b_n = b_{n-1} + b_{n-2} \qquad n \geq 2 \qquad [1]$$

$$a_n = 2b_{n-1} + b_{n-2} \qquad n \geq 2 \qquad [2]$$

with boundary conditions $b_0 = b_1 = 1$

Solving [1], assume $b_n = A\beta^n$.

Then

$$\beta^2 - \beta - 1 = 0$$

$$\beta = \frac{1 \pm \sqrt{5}}{2}$$

$$b_n = A_1\left(\frac{1 + \sqrt{5}}{2}\right)^n + A_2\left(\frac{1 - \sqrt{5}}{2}\right)^n$$

Evaluating A_1, A_2 using boundary conditions gives

$$b_n = \frac{1}{2^{n+1}\sqrt{5}}\left[(1 + \sqrt{5})^{n+1} - (1 - \sqrt{5})^{n+1}\right]$$

which, with [2], yields

$$a_n = \frac{1}{2^{n-1}\sqrt{5}}\left[(1 + \sqrt{5})^n - (1 - \sqrt{5})^n + (1 + \sqrt{5})^{n-1} - (1 - \sqrt{5})^{n-1}\right]$$

with first few values

$$a_2 = 3,\ a_3 = 5,\ a_4 = 8,\ a_5 = 13,\ \cdots$$

(b) Let a_n = number of ternary n-tuples having no adjacent 0's.

b_n = number of ternary n-tuples having no adjacent 0's and ending in 1 or 2.

Then

$$b_n = 2b_{n-1} + 2b_{n-2} \qquad n \geq 2 \qquad [1]$$

$$a_n = 3b_{n-1} + 2b_{n-2} \qquad n \geq 2 \qquad [2]$$

with boundary conditions $b_0 = 1$, $b_1 = 2$

Solving [1],

$$\beta^2 - 2\beta - 2 = 0$$

$$\beta = 1 \pm \sqrt{3}$$

$$b_n = A_1(1 + \sqrt{3})^n + A_2(1 - \sqrt{3})^n$$

Evaluating A_1, A_2, using boundary conditions

$$b_n = \frac{1}{2\sqrt{3}} [(1 + \sqrt{3})^{n+1} - (1 - \sqrt{3})^{n+1}]$$

and from [2],

$$a_n = \frac{1}{3} \{\frac{3}{2}[(1 + \sqrt{3})^n - (1 - \sqrt{3})^n] + (1 + \sqrt{3})^{n-1} - (1 - \sqrt{3})^{n-1}\}$$

with first few values

$$a_2 = 8,\ a_3 = 22,\ \cdots$$

□

3-8. The Fibonacci numbers are defined by the recurrence relation

$$f_n = f_{n-1} + f_{n-2} \qquad n \geq 2$$

together with the boundary conditions

$$f_0 = f_1 = 1$$

(a) Find the ordinary generating function $F(x) = \sum_{n=0}^{\infty} f_n x^n$ of the sequence $(f_0, f_1, f_2, \cdots)$.

(b) Let

$$P(x) = \sum_{n=0}^{\infty} f_{2n}\ x^n$$

$$Q(x) = \sum_{n=1}^{\infty} f_{2n-1} x^n$$

Find $P(x)$ and $Q(x)$.

(a)
$$\sum_{n=2}^{\infty} f_n x^n = \sum_{n=2}^{\infty} f_{n-1} x^n + \sum_{n=2}^{\infty} f_{n-2} x^n$$

$$F(x) - f_0 - f_1 x = x[F(x) - f_0] + x^2 F(x)$$

$$F(x) - 1 - x = x[F(x) - 1] + x^2 F(x)$$

$$F(x) = \frac{1}{1-x-x^2}$$

(b)

$$P(x) = \frac{F(x^{\frac{1}{2}}) + F(-x^{\frac{1}{2}})}{2} = \frac{1-x}{1-3x+x^2}$$

$$Q(x) = x^{\frac{1}{2}}\left[\frac{F(x^{\frac{1}{2}}) - F(-x^{\frac{1}{2}})}{2}\right] = \frac{x}{1-3x+x^2}$$

□

3-9. Define

$$a_n = \sum_{k=0}^{n} \binom{n+k}{2k} \qquad n \geq 1$$

$$b_n = \sum_{k=0}^{n-1} \binom{n+k}{2k+1} \qquad n \geq 1$$

and $a_0 = 1$ and $b_0 = 0$.

(a) Show that a_n and b_n satisfy the recurrence relations

$$a_{n+1} = a_n + b_{n+1} \qquad n \geq 0$$

$$b_{n+1} = a_n + b_n \qquad n \geq 0$$

(b) Find the ordinary generating functions of the sequences $(a_0,a_1,a_2,\cdots)$ and $(b_0,b_1,b_2,\cdots)$.

(c) Express a_n and b_n in terms of the Fibonacci numbers.

(a)

$$a_{n+1} = \sum_{k=0}^{n+1} \binom{n+k+1}{2k}$$

$$= \sum_{k=0}^{n+1} \left[\binom{n+k}{2k} + \binom{n+k}{2k-1}\right] \text{ using the identity } \binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}.$$

$$= \sum_{k=0}^{n} \binom{n+k}{2k} + \binom{2n+1}{2n+2} + \sum_{k=0}^{n} \binom{n+k+1}{2k+1}$$

$$= a_n + b_{n+1}$$

$$b_{n+1} = \sum_{k=0}^{n} \binom{n+k+1}{2k+1}$$

$$= \sum_{k=0}^{n-1} \left[\binom{n+k}{2k+1} + \binom{n+k}{2k}\right]$$

$$= \sum_{k=0}^{n-1} \binom{n+k}{2k+1} + \binom{2n}{2n+1} + \sum_{k=0}^{n} \binom{n+k}{2k}$$

$$= b_n + a_n$$

(b) $$\sum_{n=0}^{\infty} a_{n+1}x^n = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_{n+1}x^n$$

$$\frac{A(x) - a_0}{x} = A(x) + \frac{B(x) - b_0}{x}$$

$$\frac{A(x)-1}{x} = A(x) + \frac{B(x)}{x}$$

$$A(x) = \frac{B(x) + 1}{1-x} \qquad [1]$$

$$\sum_{n=0}^{\infty} b_{n+1}x^n = \sum_{n=0}^{\infty} b_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$\frac{B(x) - b_0}{x} = B(x) + A(x)$$

$$B(x) = \frac{x}{1-x} A(x) \qquad [2]$$

Combining [1] and [2]

$$A(x) = \frac{1-x}{1-3x+x^2}$$

$$B(x) = \frac{x}{1-3x+x^2}$$

(c) From Problem 3-8(b)

$$A(x) = P(x)$$
$$B(x) = Q(x)$$

$$a_n = f_{2n}$$

$$b_n = f_{2n-1}$$

□

3-10. Let f(n,k) denote the number of ways of selecting k numbers from the n numbers 1, 2, 3, ···, n so that no consecutive numbers will be selected.

(a) Find a recurrence relation for f(n,k).

(b) With the boundary conditions f(n,1) = n and f(n,n) = 0, show that $f(n,k) = \binom{n-k+1}{k}$ by induction.

(c) Let g(n,k) denote the number of ways of selecting k numbers from the n numbers 1, 2, 3, ···, n so that no consecutive numbers will be selected. Here, n and 1 are also taken as consecutive numbers. Express g(n,k) in terms of the f(n,k)'s and use the result in part (b) to find g(n,k).

(a) $f(n,k) = \underbrace{f(n-1,k)}_{\text{exclude "1"}} + \underbrace{f(n-2,k-1)}_{\text{include "1"}}$

(b) **Basis**

$f(n,1) = n = \binom{n-1+1}{1}$ $\quad n \geq 1$

$f(n,n) = 0 = \binom{n-n+1}{n}$ $\quad n \geq 2$

Induction Hypothesis

Assume $f(a,b) = \binom{a-b+1}{b}$ for $b \leq a \leq n$

Induction Step

Using the recurrence relation,

$$f(n+1,k) = f(n,k) + f(n-1,k-1)$$
$$= \binom{n-k+1}{k} + \binom{n-k+1}{k-1}$$
$$= \binom{n-k+2}{k}$$
$$= \binom{(n+1)-k+1}{k}$$

(c) $g(n,k) = \underbrace{f(n-1,k)}_{\substack{\text{exclude a} \\ \text{special integer}}} + \underbrace{f(n-3,k-1)}_{\substack{\text{include that} \\ \text{special integer}}}$

From (b),

$g(n,k) = \binom{n-k}{k} + \binom{n-k-1}{k-1}$.

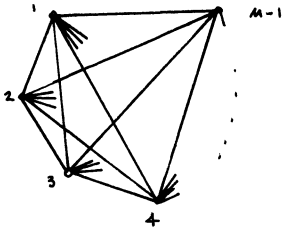
□

3-11. Let f_n denote the number of non-overlapping regions into which the interior of a convex n-gon is divided by its diagonals. Suppose that no three diagonals meet at one point.

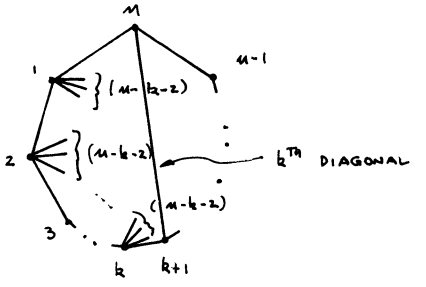
(a) Show that $f_n - f_{n-1} = \frac{(n-1)(n-2)(n-3)}{6} + n-2 \quad n \geq 3$

(b) Let $F(x) = \sum_{n=0}^{\infty} f_n x^n$, and let $f_0 = f_1 = f_2 = 0$. Find F(x) and thus an expression for f_n.

(a) Consider a convex (n-1)-gon with all diagonals drawn and vertices numbered 1,2 1, 2, $\cdots$, (n-1) as follows:



Add the n^{th} vertex and two edges to vertices 1 and (n-1), thereby creating one new region. Let the k^{th} diagonal from vertex n be incident with vertex k+1, k = 1,2,$\cdots$,(n-3).



The k^{th} diagonal intersects k(n-k-2) other diagonals and so is divided into k(n-k-2)+1 segments. Each segment divides a region into two regions, so that k(n-k-2) new regions are created by the k^{th} diagonal.

So,

$$f_n - f_{n-1} = \sum_{k=1}^{n-3} [k(n-k-2) + 1] + 1 \qquad n \geq 3$$

$$= (n-2) \sum_{k=1}^{n-3} k - \sum_{k=1}^{n-3} k^2 + (n-3) + 1$$

$$= \frac{(n-2)(n-2)(n-3)}{2} - [\binom{n-1}{3} + \binom{n-2}{3}] + n-2$$

$$= \binom{n-1}{3} + n-2 \qquad n \geq 3$$

(b) $$\sum_{n=3}^{\infty} f_n x^n - \sum_{n=3}^{\infty} f_{n-1} x^n = \sum_{n=3}^{\infty} \frac{(n-1)(n-2)(n-3)}{6} x^n + \sum_{n=3}^{\infty} (n-2)x^n$$

$$F(x) - f_0 - f_1 x - f_2 x^2 - x[F(x) - f_0 - f_1 x] = \frac{x^4}{(1-x)^4} + [\frac{x}{(1-x)^2} - x - 2x^2] - 2[\frac{x}{1-x} - x - x^2]$$

$$F(x)\ (1-x) = \frac{x^3-x^4+x^5}{(1-x)^4}$$

$$F(x) = \frac{x^3-x^4+x^5}{(1-x)^5} = (x^3 - x^4 + x^5) \sum_{r=0}^{\infty} \binom{r+4}{4} x^r$$

$$f_n = \binom{n+1}{4} - \binom{n}{4} + \binom{n-1}{4} \qquad n \geq 3$$

□

3-12. The Bernoulli numbers $b_0, b_1, b_2, \cdots$ are defined by the recurrence relation

$$b_n = \sum_{k=0}^{n} \binom{n}{k} b_{n-k} \qquad n \geq 1$$

together with the boundary condition

$$b_0 = 1$$

(a) Compute b_0, b_1, b_2, b_3, b_4, and b_5.

(b) Show that the exponential generating function of the sequence $(b_0, b_1, b_2, \cdots)$ is $x/(e^x-1)$.

(a)

$$b_n = b_n + \sum_{k=1}^{n} \binom{n}{k} b_{n-k} \qquad n \geq 2$$

or,

$$\sum_{k=1}^{n} \binom{n}{k} b_{n-k} = 0 \qquad n \geq 2$$

$b_0 = 1$

n = 2, $2b_1 + b_0 = 0, \quad b_1 = -\frac{1}{2}$

n = 3, $3b_2 + 3b_1 + b_0 = 0, \quad b_2 = \frac{1}{6}$

n = 4, $4b_3 + 6b_2 + 4b_1 + b_0 = 0, \quad b_3 = 0$

n = 5, $5b_4 + 10b_3 + 10b_2 + 5b_1 + b_0 = 0, \quad b_4 = -\frac{1}{30}$

n = 6, $6b_5 + 15b_4 + 20b_3 + 15b_2 + 6b_1 + b_0 = 0, \quad b_5 = 0$

(b) $$B(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}$$

$$B(x) - b_0 - b_1 x = \sum_{n=2}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} b_{n-k} \right) \frac{x^n}{n!}$$

$$= \sum_{n=2}^{\infty} \underbrace{\left(\sum_{k=0}^{n} b_{n-k} \frac{x^{n-k}}{(n-k)!} \cdot \frac{x^k}{k!} \right)}_{\text{This is the term } c_n x^n \text{ in } C(x) = B(x) \cdot e^x}$$

$$= B(x) \cdot e^x - [b_0 + (b_0 + b_1)x]$$

$$B(x) - 1 + \frac{x}{2} = B(x) \cdot e^x - 1 - \frac{x}{2}$$

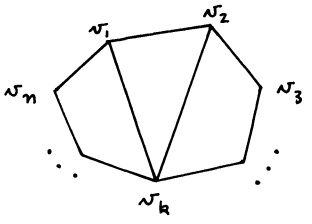
$$B(x) = \frac{x}{e^x - 1}$$

□

3-13. In how many ways can a convex n-gon be divided into triangles by non-intersecting diagonals?

Let a_n be the number of ways a convex n-gon may be divided into triangles by non-intersecting diagonals.

Consider a convex n-gon with vertices labeled v_1, v_2, $\cdots$, v_n as shown below.



In any division into triangles, vertices v_1 and v_2 are corners of some triangle, along with another vertex v_k, $3 \leq k \leq n$.

If $k = 3$, The vertices $v_1, v_3, v_4, \cdots, v_n$ form a convex (n-1)-gon. There are a_{n-1} ways to complete the division into triangles.

If $3 < k < n$, the vertices $v_2, v_3, \cdots, v_k$ form a convex (k-1)-gon, and the vertices $v_1, v_k, v_{k+1}, \cdots, v_n$ form a convex (n-k+2)-gon. There are $a_{k-1} \cdot a_{n-k+2}$ ways to complete the division into triangles.

If k = n, the vertices $v_2, v_3, \cdots, v_n$ form a convex (n-1)-gon. There are a_{n-1} ways to complete the division into triangles.

So,

$$a_n = a_{n-1} + a_3a_{n-2} + a_4a_{n-3} + \cdots + a_{n-3}a_4 + a_{n-2}a_3 + a_{n-1} \qquad n \geq 5$$

with boundary conditions

$$a_3 = 1, \; a_4 = 2$$

By choosing

$$a_0 = a_1 = 0, \quad a_2 = 1$$

the recurrence relation may be rewritten as

$$a_n = a_0a_{n+1} + a_1a_n + a_2a_{n-1} + a_3a_{n-2} + \cdots + a_{n-2}a_3 + a_{n-1}a_2 + a_na_1 + a_{n+1}a_0 \qquad n \geq 3$$

Solving,

$$\sum_{n=3}^{\infty} a_n x^{n+1} = \sum_{n=3}^{\infty} (a_0a_{n+1} + a_1a_n + \cdots + a_na_1 + a_{n+1}a_0)x^{n+1}$$

$$x[A(x) - x^2] = A(x) \cdot A(x) - [(a_0a_0) + (a_0a_1 + a_1a_0)x + (a_0a_2 + a_1a_1 + a_2a_0)x^2 + (a_0a_3 + a_1a_2 + a_2a_1 + a_3a_0)x^3]$$

$$A^2(x) - x\,A(x) + x^3 = 0$$

$$A(x) = \frac{x \pm x\sqrt{1-4x}}{2}$$

$$(1-4x)^{\frac{1}{2}} = 1 + \sum_{r=1}^{\infty} \frac{(\frac{1}{2})(-\frac{1}{2}) \cdots (\frac{1}{2} - r + 1)}{r!} (-1)^r 2^r 2^r x^r$$

$$= 1 - \sum_{r=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2r-3) \cdot r! \cdot 2^r}{r!r!} x^r$$

$$= 1 - \sum_{r=1}^{\infty} \frac{(2r)!}{r!r!\,(2r-1)} x^r$$

$$= 1 - \sum_{r=1}^{\infty} \frac{\binom{2r}{r}}{2r-1} x^r$$

Choosing the negative sign in the expression for A(x),

$$A(x) = \frac{x}{2} - \frac{x}{2}\left[1 - \sum_{r=1}^{\infty} \frac{\binom{2r}{r}}{2r-1} x^r\right]$$

$$= \frac{1}{2} \sum_{r=1}^{\infty} \frac{\binom{2r}{r}}{2r-1} x^{r+1}$$

$$= \frac{1}{2} \sum_{r=2}^{\infty} \frac{\binom{2r-2}{r-1}}{2r-3} x^r$$

$$a_n = \begin{cases} 0 & n = 0,1 \\ 1 & n = 2 \\ \frac{1}{2} \frac{\binom{2n-2}{n-1}}{2n-3} & n \geq 3 \end{cases}$$

□

3-14. When a coin is tossed n times all possible outcomes can be represented by sequences of heads (H) and tails (T) of length n. A sequence of H's and T's contains as many patterns of HHTHH as there are non-overlapping sequences of HHTHH. The pattern HHTHH is said to occur at the nth toss, if the nth toss adds a new HHTHH pattern to the sequence. A sequence of length n is acceptable if the HHTHH pattern occurs at the nth toss. Let u_n be the number of acceptable sequences of length n.

(a) Find a recurrence relation for u_n.

(b) Let $u_0 = 1$, and let $U(x) = \sum_{n=0}^{\infty} u_n x^n$. Find U(x).

(c) Let f_n be the number of sequences in which the pattern HHTHH occurs for the first time at the nth toss. If $f_0 = 0$, find the ordinary generating function F(x) of the sequences $(f_0, f_1, f_2, \cdots)$.

(a) $\underbrace{2^{n-5}}_{\substack{\text{\# of n-toss} \\ \text{sequences ending} \\ \text{with HHTHH}}} = \underbrace{u_n}_{\substack{\text{\# in which HHTHH} \\ \text{occurs at } n^{th} \text{ toss}}} + \underbrace{u_{n-3} + u_{n-4}}_{\substack{\text{\# in which} \\ \text{HHTHH does not} \\ \text{occur at } n^{th} \text{ toss}}} \qquad n \geq 5$

(b) $$\sum_{n=5}^{\infty} u_n x^n + \sum_{n=5}^{\infty} u_{n-3} x^n + \sum_{n=5}^{\infty} u_{n-4} x^n = \sum_{n=5}^{\infty} 2^{n-5} x^n$$

$$U(x) - u_0 - u_1x - u_2x^2 - u_3x^3 - u_4x^4 + x^3[U(x) - u_0 - u_1x] + x^4[U(x) - u_0] = \frac{x^5}{1-2x}$$

Choose $u_1 = u_2 = u_3 = u_4 = 0$

$$U(x) - 1 + x^3[U(x) - 1] + x^4[U(x) - 1] = \frac{x^5}{1-2x}$$

$$U(x) = 1 + \frac{x^5}{(1-2x)(1+x^3+x^4)}$$

(c) Using Equation 3-9,

$$F(x) = 1 - \frac{1}{U(x)} = \frac{x^5}{1-2x+x^3-x^4-x^5}$$

□

3-15. Find the number of n-digit binary sequences in which the pattern 111 occurs exactly twice with the second occurrence at the suffix. For example, the first and the fourth sequences in the following satisfy the condition, whereas the second and the third sequences do not.

00 111 11 010 111
00 111 111 0 111 1
00 110 110 111 11
0 110 110 111 111

Let a_n be the number of n-digit binary sequences in which "111" occurs exactly twice with the second occurrence at the suffix.

Let b_n be the number of n-digit binary sequences in which "111" occurs for the first time at the n^{th} digit.

$$a_n = b_3b_{n-3} + b_4b_{n-4} + \cdots + b_{n-3}b_3 \qquad n \geq 6$$

$$\sum_{n=6}^{\infty} a_n x^n = \sum_{n=6}^{\infty} (b_3b_{n-3} + b_4b_{n-4} + \cdots + b_{n-3}b_3)x^n$$

Choose $a_0 = a_1 = a_2 = a_3 = a_4 = a_5 = 0$

$b_0 = b_1 = b_2 = 0$

$A(x) = B(x)\,B(x)$

Let c_n be the number of n-digit binary sequences in which "111" occurs at the n^{th} digit.

$$\underbrace{2^{n-3}}_{\#\text{ of } n\text{-digit sequences ending with "111"}} = \underbrace{c_n}_{\#\text{ in which "111" occurs at } n^{th} \text{ digit}} + \underbrace{c_{n-1} + c_{n-2}}_{\#\text{ in which "111" does not occur at } n^{th} \text{ digit}} \qquad n \geq 3$$

$$\sum_{n=3}^{\infty} c_n x^n + \sum_{n=3}^{\infty} c_{n-1} x^n + \sum_{n=3}^{\infty} c_{n-2} x^n = \sum_{n=3}^{\infty} 2^{n-3} x^n$$

$$C(x) - 1 + x[C(x) - 1] + x^2[C(x) - 1] = \frac{x^3}{1-2x} \qquad \text{where } c_0 = 1,\ c_1 = c_2 = 0$$

$$C(x) = 1 + \frac{x^3}{(1-2x)(1+x+x^2)}$$

Using Equation 3-9,

$$B(x) = 1 - \frac{1}{C(x)} = \frac{x^3}{1-x-x^2-x^3}$$

Finally,

$$A(x) = B(x)\,B(x) = \frac{x^6}{1-2x-x^2+3x^4+2x^5+x^6} = x^6 + 2x^7 + 5x^8 + 12x^9 + \cdots$$

□

3-16. Two players, A and B, gamble by tossing a coin which has probabilities p and q to turn up head and tail, respectively. (p + q = 1). Initially, A has d dollars and B has t - d dollars. At each toss, A wins a dollar if the head shows and loses a dollar if the tail shows. The game lasts as long as neither of the players is broke. Let f(d,n) be the probability that A goes broke as a result of the nth toss.

(a) Find a recurrence relation for f(d,n). Also, determine the boundary conditions f(0,n), f(t,n), and f(d,0).

(b) Let $F_d(x) = \sum_{n=0}^{\infty} f(d,n)x^n$. Find $F_0(x)$, $F_t(x)$, and a recurrence relation for $F_d(x)$.

(c) Solve the recurrence relation found in part (b) for $F_d(x)$.

(a) Probability {A loses d dollars in n tosses} =

Probability { A wins first toss and loses (d+1) dollars in (n-1) tosses} +

Probability {A loses first toss and loses (d-1) dollars in (n-1) tosses}

$$f(d,n) = pf(d+1, n-1) + q\,f(d-1, n-1)$$

$$f(0,n) = \begin{cases} 0 & n > 0 \\ 1 & n = 0 \end{cases}$$

$$f(t,n) = 0 \qquad n \geq 0$$

$$f(d,0) = 0 \qquad d > 0$$

(b) $F_0(x) = 1$

$F_t(x) = 0$

$$F_d(x) = \sum_{n=1}^{\infty} f(d,n)x^n = p \sum_{n=1}^{\infty} f(d+1,\ n-1)x^n + q \sum_{n=1}^{\infty} f(d-1,\ n-1)x^n$$

$$= px\, F_{d+1}(x) + qx\, F_{d-1}(x)$$

(c) Let $F_d(x) = A(x)\ \alpha(x)^d$

$px\ \alpha(x)^2 - \alpha(x) + qx = 0$

$$\alpha(x) = \frac{1 \pm \sqrt{1-4pqx^2}}{2px}$$

Let $\alpha_1(x) = \dfrac{1 + \sqrt{1-4pqx^2}}{2px}$

$$\alpha_2(x) = \frac{1 - \sqrt{1-4pqx^2}}{2px}$$

$$F_d(x) = A_1(x)\ \alpha_1(x)^d + A_2(x)\ \alpha_2(x)^d$$

The boundary conditions

$$F_0(x) = A_1(x) + A_2(x) = 1$$

$$F_t(x) = A_1(x)\alpha_1(x)^t + A_2(x)\alpha_2(x)^t = 0$$

give

$$A_1(x) = \frac{-\alpha_2(x)^t}{\alpha_1(x)^t - \alpha_2(x)^t}$$

$$A_2(x) = \frac{\alpha_1(x)^t}{\alpha_1(x)^t - \alpha_2(x)^t}$$

and

$$F_d(x) = \alpha_1(x)^d\ \alpha_2(x)^d\ \left(\frac{\alpha_1(x)^{t-d} - \alpha_2(x)^{t-d}}{\alpha_1(x)^t - \alpha_2(x)^t}\right)$$

$$= \left(\frac{q}{p}\right)^d \left(\frac{\alpha_1(x)^{t-d} - \alpha_2(x)^{t-d}}{\alpha_1(x)^t - \alpha_2(x)^t}\right)$$

□

3-17. (a) Let u_n be the number of binary sequences of length n in which a run of r 1's occurs at the nth digit. Find a recurrence relation for u_n.

(b) Find the ordinary generating function U(x) of the sequences $(u_0, u_1, u_2, \cdots)$.

(c) Let v_n be the number of binary sequences of length n in which a run of p 0's occurs at the nth digit. Find the ordinary generating function V(x) of the sequence $(v_0, v_1, v_2, \cdots)$.

(d) Let w_n be the number of binary sequences of length n in which either a run of r 1's or a run of p 0's occurs at the nth digit. Find the ordinary generating function W(x) of the sequence $(w_0, w_1, w_2, \cdots)$.

(e) Let t_n be the number of binary sequences of length n in which a run of p 0's occurs before a run of r 1's. Find the ordinary generating function T(x) of the sequence $(t_0, t_1, t_2, \cdots)$.

(a) $\underbrace{2^{n-r}}_{\text{\# of sequences ending in r 1's}} = \underbrace{u_n}_{\text{\# in which a run occurs at the } n^{th} \text{ digit}} + \underbrace{u_{n-1} + u_{n-2} + \cdots + u_{n-r+1}}_{\text{\# in which a run does not occur at the } n^{th} \text{ digit}} \qquad n \geq r$

(b) $$\sum_{n=r}^{\infty} 2^{n-r} x^n = \sum_{n=r}^{\infty} u_n x^n + \sum_{n=r}^{\infty} u_{n-1} x^n + \sum_{n=r}^{\infty} u_{n-2} x^n + \cdots + \sum_{n=r}^{\infty} u_{n-r+1} x^n$$

Let $u_0 = 1$, $u_1 = u_2 = \cdots = u_{r-1} = 0$

$$\frac{x^r}{1-2x} = [U(x) - 1] + x[U(x) - 1] + x^2[U(x) - 1] + \cdots + x^{r-1}[U(x) - 1]$$

$$U(x) = 1 + \frac{x^r}{(1-2x)(1+x+x^2+\cdots+x^{r-1})}$$

$$= 1 + \frac{x^r(1-x)}{(1-2x)(1-x^r)}$$

(c) Similarly

$$V(x) = 1 + \frac{x^p(1-x)}{(1-2x)(1-x^p)}$$

(d) $W(x) = U(x) + V(x) - 1$

(e) Let a_n be the number of n-digit sequences in which a run of r 1's occurs at the n^{th} digit for the first time.

Let b_n be the number of n-digit sequences in which a run of p 0's occurs at the n^{th} digit for the first time.

Using Equation 3-9,

$$A(x) = 1 - \frac{1}{U(x)}$$

$$B(x) = 1 - \frac{1}{V(x)}$$

Indexing by the digit positions at which the run of 0's and the run of 1's occur for the first time,

$$t_n = b_p(a_r \cdot 2^{n-p-r} + a_{r+1} \cdot 2^{n-p-r-1} + a_{r+2} \cdot 2^{n-p-r-2} + \cdots + a_{n-p}) +$$

$$b_{p+1}(a_r \cdot 2^{n-p-r-1} + a_{r+1} \cdot 2^{n-p-r-2} + \cdots + a_{n-p-1}) +$$

$$\vdots$$

$$b_{n-p} \cdot a_r$$

$$T(x) = A(x)\, B(x) \cdot \frac{1}{1-2x}$$

$$= (1 - \frac{1}{U(x)})(1 - \frac{1}{V(x)})(\frac{1}{1-2x})$$

□

3-18. A coin is tossed 2n times. How many of the 2^{2n} outcomes have the number of heads and tails equalized for the first time after 2n tosses?

Let a_n = The number of length - 2n sequences in which the number of heads and tails is equalized for the first time after 2n tosses

b_n = The number of length - 2n sequences in which the number of heads and tails is equalized after 2n tosses

$$b_n = \binom{2n}{n}$$

$$B(x) = 1 + \binom{2}{1}x + \binom{4}{2}x^2 + \cdots + \binom{2k}{k}x^k + \cdots$$

$$= (1 - 4x)^{-\frac{1}{2}} \qquad \text{(see Example 2-5)}$$

$$b_n = a_n b_0 + a_{n-1}b_1 + a_{n-2}b_2 + \cdots + a_1 b_{n-1} \qquad \text{for } n \geq 1$$

Choosing $a_0 = 0$,

$$B(x) - 1 = A(x)\,B(x)$$

$$A(x) = 1 - \frac{1}{B(x)} = 1 - (1-4x)^{\frac{1}{2}}$$

$$(1-4x)^{\frac{1}{2}} = 1 + \sum_{r=1}^{\infty} \frac{(\frac{1}{2})(-\frac{1}{2}) \cdots (\frac{1}{2}-r+1)}{r!} (-4)^r x^r$$

$$= 1 - \sum_{r=1}^{\infty} \frac{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \cdots \cdot \frac{2r-3}{2}}{r!} 2^r 2^r x^r$$

$$= 1 - \sum_{r=1}^{\infty} \frac{1 \cdot 1 \cdot 3 \cdot \cdots \cdot (2r-3) \cdot 2^r r!}{r!r!} x^r$$

$$= 1 - \sum_{r=1}^{\infty} \frac{(2r)!}{(2r-1)\, r!r!} x^r$$

$$= 1 - \sum_{r=1}^{\infty} \frac{1}{2r-1} \binom{2r}{r} x^r$$

$$A(x) = \sum_{r=1}^{\infty} \frac{1}{2r-1} \binom{2r}{r} x^r \qquad a_n = \frac{1}{2n-1} \binom{2n}{n}$$

□

3-19. (a) Find the number of binary sequences of length n that have the pattern 0101 occurring at the nth digit and have an even number of 0's. Also find the number of those that have an odd number of 0's.

(b) Find the number of binary sequences of length n that have the first occurence of the pattern 0101 at the nth digit and have an even number of 0's. Also find the number of those that have an odd number of 0's.

(a) Let $a_n (b_n)$ be the number of binary n-tuples that have the pattern 0101 occurring at the n^{th} digit, and contain an even (odd) number of 0's.

$$\frac{1}{2} \cdot 2^{n-4} = a_n + b_{n-2} \qquad n \geq 5$$

$$\frac{1}{2} \cdot 2^{n-4} = b_n + a_{n-2} \qquad n \geq 5$$

with boundary conditions,

$$a_0 = a_1 = a_2 = a_3 = 0,\ a_4 = 1$$

$$b_0 = b_1 = b_2 = b_3 = b_4 = 0$$

$$\sum_{n=5}^{\infty} 2^{n-5}x^n = \sum_{n=5}^{\infty} a_n x^n + \sum_{n=5}^{\infty} b_{n-2}x^n$$

$$\frac{x^5}{1-2x} = A(x) - x^4 + x^2 B(x) \qquad [1]$$

$$\sum_{n=5}^{\infty} 2^{n-5}x^n = \sum_{n=5}^{\infty} b_n x^n + \sum_{n=5}^{\infty} a_{n-2}x^n$$

$$\frac{x^5}{1-2x} = B(x) + x^2 A(x) \qquad [2]$$

Solving [1] and [2],

$$A(x) = \frac{x^4 - x^5 - x^7}{(1-2x)(1-x^4)}$$

$$B(x) = \frac{x^5 - x^6 + x^7}{(1-2x)(1-x^4)}$$

(b) Let c_n (d_n) be the number of binary n-tuples that have the pattern 0101 occurring for the first time at the n^{th} digit and contain an even (odd) number of 0's.

$$a_n = c_n + c_{n-4}a_4 + c_{n-5}a_5 + c_{n-6}a_6 + \cdots + c_4 a_{n-4} + d_{n-5}b_5 + d_{n-6}b_6 + \cdots + d_5 b_{n-5} \qquad n \geq 5$$

Choose $c_0 = c_1 = c_2 = c_3 = 0,\ c_4 = 1$

$$d_0 = d_1 = d_2 = d_3 = d_4 = 0$$

$$A(x) = C(x)[A(x) + 1] + D(x)\,B(x) \qquad [3]$$

$$b_n = d_n + d_{n-4}a_4 + d_{n-5}a_5 + \cdots + d_5 a_{n-5} + c_{n-5}b_5 + c_{n-6}b_6 + \cdots + c_4 b_{n-4} \qquad n \geq 5$$

$$B(x) = D(x)[A(x) + 1] + C(x)\,B(x) \qquad [4]$$

Solving the implicit relations [3] and [4],

$$C(x) = \frac{A(x)\,[A(x) + 1] - B^2(x)}{[A(x) + 1]^2 - B^2(x)}$$

$$D(x) = \frac{B(x)}{[A(x) + 1]^2 - B^2(x)}$$

□

3-20. A sequence of binary digits is fed to a counter at the rate of 1 digit/sec. The counter is designed to register 1's in the input sequence. However, it is so slow that it is locked for exactly p sec. following each registration, during which time input digits are ignored.

(a) How many input sequences or length n would cause the counter to register exactly r 1's, finishing in the unlocked state.

(b) Let f_n be the number of binary sequences of length n at the end of which the counter is unlocked. Find a recurrence relation which f_n satisfies, and find the ordinary generating function of the sequence $(f_0, f_1, f_2, \cdots)$.

(a) $\binom{n-rp}{r} \cdot 2^{rp}$

(b) $f_n = f_{n-1} + 2^p f_{n-p-1} \qquad n \geq p+1$

$$\sum_{n=p+1}^{\infty} f_n x^n = \sum_{n=p+1}^{\infty} f_{n-1} x^n + 2^p \sum_{n=p+1}^{\infty} f_{n-p-1} x^n$$

Let $f_0 = f_1 = f_2 = \cdots = f_p = 1$

$$F(x) - 1 - x - x^2 - \cdots - x^p = x[F(x) - 1 - x - x^2 - \cdots - x^{p-1}] + 2^p x^{p+1} F(x)$$

$$F(x) = \frac{1}{1 - x - 2^p x^{p+1}}$$

□

3-21. Find the number of binary sequences of length n that have exactly one pair of consecutive 0's.

Let $\begin{bmatrix} a_n \\ b_n \\ c_n \\ d_n \end{bmatrix}$ be the number of binary n-tuples in which the pattern 00

pattern 00 $\begin{bmatrix} \text{occurs at the } n^{th} \text{ digit} \\ \text{occurs for the first time at the } n^{th} \text{ digit} \\ \text{does not occur} \\ \text{occurs exactly once} \end{bmatrix}$.

$$2^{n-2} = a_n + a_{n-1} \qquad n \geq 2$$

$$\sum_{n=2}^{\infty} 2^{n-2} x^n = \sum_{n=2}^{\infty} a_n x^n + \sum_{n=2}^{\infty} a_{n-1} x^n$$

$$\frac{x^2}{1-2x} = A(x) - a_0 - a_1 x + x[A(x) - a_0]$$

Let $a_0 = 1,\ a_1 = 0$

$$A(x) = 1 + \frac{x^2}{(1-2x)(1+x)}$$

$$B(x) = 1 - \frac{1}{A(x)} = \frac{x^2}{1-x-x^2}$$

Let c_n^1 and c_n^0 be the number of binary n-tuples ending in 1 and 0 respectively in which the pattern 00 does not occur.

$$c_n = c_n^1 + c_n^0$$

$$c_n^1 = c_{n-1}^1 + c_{n-1}^0$$

$$c_n^0 = c_{n-1}^1$$

$$c_n^1 = c_{n-1}^1 + c_{n-2}^1$$

$$c_n = c_n^1 + c_{n-1}^1$$

$$\sum_{n=2}^{\infty} c_n^1 x^n = \sum_{n=2}^{\infty} c_{n-1}^1 x^n + \sum_{n=2}^{\infty} c_{n-2}^1 x^n$$

$$C^1(x) - c_0^1 - c_1^1 x = x[C^1(x) - c_0^1] + x^2\, C^1(x)$$

Let $c_0^1 = 1,\ c_1^1 = 1$

$$C^1(x) = \frac{1}{1-x-x^2}$$

$$\sum_{n=1}^{\infty} c_n x^n = \sum_{n=1}^{\infty} c_n^1 x^n + \sum_{n=1}^{\infty} c_{n-1}^1 x^n$$

$$C(x) - c_0 = C^1(x) - c_0^1 + x\, C^1(x)$$

Let $c_0 = 1$

$$C(x) = \frac{1+x}{1-x-x^2}$$

Finally

$$d_n = b_n + b_{n-1}c_0 + b_{n-2}c_1 + b_{n-3}c_2 + \cdots + b_2 c_{n-3} \qquad n \geq 2$$

$$D(x) = B(x)[x\, C(x) + 1]$$

$$= \frac{x^2}{(1-x-x^2)^2} = \frac{x^2}{1-2x-x^2+2x^3+x^4}$$

$$= x^2 + 2x^3 + 5x^4 + 10x^5 + 20x^6 + \cdots$$

□

3-22. (a) Recall that there are $S(r,n)$ ways to distribute r distinct objects into n non-distinct cells with no cell left empty, where $S(r,n)$ is the Stirling number of the second kind. Using combinatorial arguments, show that $S(r,n)$ satisfies the following recurrence relation:

$$S(r + 1, n) = S(r,n - 1) + nS(r,n) \qquad n \geq 1$$

(b) Let $G_n(x)$ be the ordinary generating function of the sequence $(S(0,n), S(1,n), S(2,n), \cdots)$; that is, $G_n(x) = \sum_{r=0}^{\infty} S(r,n)x^r$. Find a recurrence relation satisfied by $G_n(x)$. The boundary conditions are

$$S(r,n) = \begin{cases} 1 & r = 0,\ n = 0 \\ 0 & r = 0,\ n > 0 \quad \text{or} \quad r > 0, n = 0 \end{cases}$$

(c) From the recurrence relation in part (b) find $G_n(x)$. [Do not solve the recurrence relation; try to obtain the general form for $G_n(x)$ by observation.]

(d) Define

$$E_n(x) = \sum_{r=0}^{\infty} S(r,n)\frac{x^r}{r!}$$

which is the exponential generating function of the sequence $(S(0,n),S(1,n),S(2,n), \cdots)$. Show that

$$\frac{d}{dx} E_n(x) - nE_n(x) = E_{n-1}(x) \qquad n \geq 1$$

Single out a special object, **0**.

(a) $\underbrace{S(r+1, n)}$ = $\underbrace{S(r, n-1)}$ + $\underbrace{nS(r,n)}$ $\qquad n \geq 1$

$r+1$ distinct objects into n non-distinct cells

0 is alone in a cell: r distinct objects into n-1 non-distinct cells, 0 into remaining cell

0 is not alone in a cell: r distinct objects into n non-distinct cells, 0 into one of these.

(b) $$\sum_{r=0}^{\infty} S(r+1, n)x^r = \sum_{r=0}^{\infty} S(r,n-1)x^r + n\sum_{r=0}^{\infty} S(r,n)x^r$$

$$\frac{1}{x}[G_n(x) - S(0,n)] = G_{n-1}(x) + n\,G_n(x)$$

$$S(0,n) = 0 \qquad n \geq 1$$

$$G_n(x) = \frac{x}{1-nx}\, G_{n-1}(x) \qquad n \geq 1$$

(c) $G_0(x) = 1$

$$G_1(x) = \frac{x}{1-x}$$

$$G_2(x) = \frac{x^2}{(1-x)(1-2x)}$$

$$\vdots$$

$$G_n(x) = \frac{x^n}{(1-x)(1-2x)\cdots(1-nx)}$$

d) $$\sum_{r=0}^{\infty} S(r+1, n)\frac{x^r}{r!} - n\sum_{r=0}^{\infty} S(r,n)\frac{x^r}{r!} = \sum_{r=0}^{\infty} S(r, n-1)\frac{x^r}{r!} \qquad n \geq 1$$

$$\frac{d}{dx}E_n(x) - n\,E_n(x) = E_{n-1}(x) \qquad n \geq 1$$

□

3-23. Let $s(r,n)$ denote the Stirling number of the first kind which is defined by the following equation, where x is a formal variable.

$$\sum_{n=0}^{r} s(r,n)x^n = x(x-1)\cdots(x-r+1)$$

(a) Show that $s(r,n)$ satisifes the recurrence relation

$s(r+1, n) = s(r, n-1) - r\cdot s(r,n)$

(b) If $E_n(x) = \sum_{r=0}^{\infty} s(r,n)\frac{x^r}{r!}$ is the exponential generating function of the sequence $(s(0,n), s(1,n), s(2,n), \cdots)$, show that $E_n(x)$ satisfies the differential equation

$$(1+x)\frac{d}{dx}E_n(x) = E_{n-1}(x)$$

(c) Find $E_n(x)$. The boundary conditions are $s(0,0) = 1$ and $s(r,0) = 0$ for $r \neq 0$.

(a) $$\sum_{n=0}^{r} s(r,n)x^n = x(x-1)(x-2)\cdots(x-r+1)$$

$$\sum_{n=0}^{r} s(r+1, n)x^n = x(x-1)(x-2)\cdots(x-r+1)(x-r)$$

$$\sum_{n=0}^{r+1} s(r+1, n)x^n = (x-r)\sum_{n=0}^{r} s(r,n)x^n$$

$$= \sum_{n=0}^{r} s(r,n)x^{n+1} - r\sum_{n=0}^{r} s(r,n)x^n$$

$$= \sum_{n=1}^{r+1} s(r,n-1)x^n - r \sum_{n=0}^{r} s(r,n)x^n$$

$$s(r+1,\ n) \quad = \quad s(r,\ n-1) - r\cdot s(r,n) \qquad \text{for} \quad 1 \le n \le r$$

(b)
$$\frac{d}{dx} E_n(x) = \sum_{r=1}^{\infty} s(r,n) \frac{x^{r-1}}{(r-1)!}$$

$$(1+x) \frac{d}{dx} E_n(x) = \sum_{r=1}^{\infty} s(r,n) \frac{x^{r-1}}{(r-1)!} + \sum_{r=1}^{\infty} s(r,n) \frac{x^r}{(r-1)!}$$

$$E_{n-1}(x) = \sum_{r=0}^{\infty} s(r,n-1) \frac{x^r}{r!} = \sum_{r=0}^{\infty} s(r+1,n) \frac{x^r}{r!} + \sum_{r=0}^{\infty} r\cdot s(r,n) \frac{x^r}{r!}$$

$$= \sum_{r=1}^{\infty} s(r,n) \frac{x^{r-1}}{(r-1)!} + \sum_{r=1}^{\infty} s(r,n) \frac{x^r}{(r-1)!}$$

(c)
$$E_0(x) = 1$$

$$(1+x) \frac{d}{dx} E_1(x) = 1$$

$$E_1(x) = \ell n(1+x)$$

$$(1+x) \frac{d}{dx} E_2(x) = \ell n(1+x)$$

$$E_2(x) = \frac{[\ell n(1+x)]^2}{2!}$$

$$\vdots$$

$$E_n(x) = \frac{[\ell n(1+x)]^n}{n!}$$

4-1. In how many ways can three 0's, three 1's, and three 2's be arranged so that no three adjacent digits are the same in an arrangement?

Define the following properties of an arrangement:

a_0 - 3 0's appear together
a_1 - " 1's " "
a_2 - " 2's " "

$$N(a_0'a_1'a_2') = \frac{9!}{3!3!3!} - 3\left(\frac{7!}{3!3!}\right) + 3\left(\frac{5!}{3!}\right) - 3! = 1314$$

□

4-2. A man has six friends. He has met each of them at dinner 12 times, every two of them six times, every three of them four times, every four of them three times, every five twice, and all six only once. He has dined out eight times without meeting any of them. How many times has he dined out altogether?

Let a_i, $i = 1, 2, \cdots, 6$, be the property of a dinner that friend i was there.

$$N(a_1'a_2'\cdots a_6') = 8 = N - \binom{6}{1}\cdot 12 + \binom{6}{2}\cdot 6 - \binom{6}{3}\cdot 4 + \binom{6}{4}\cdot 3 - \binom{6}{5}\cdot 2 + \binom{6}{6}\cdot 1$$

$$= N - 28$$

$$N = 36$$

□

4-3. A symmetric expression in three variables x, y, and z contains nine terms. Four terms contain the variable x. Two terms contain the variables x, y, and z. One term is a constant. How many terms contain the variables x and y?

Define the following properties of a term:

a_x - contains the variable x
a_y - " " " y
a_z - " " " z

From the problem statement,

$$N(a_x) = 4$$
$$N(a_x a_y a_z) = 2$$
$$N(a_x'a_y'a_z') = 1$$
$$N = 9$$

and because the function is symmetric,

$$N(a_x) = N(a_y) = N(a_z)$$

$$N(a_x a_y) = N(a_x a_z) = N(a_y a_z)$$

$$N(a_x' a_y' a_z') = 1 = 9 - 3 \cdot 4 + 3 \cdot N(a_x a_y) - 2$$

$$N(a_x a_y) = 2$$

□

4-4. Find the number of binary sequences of length 5 in which every 1 is adjacent to another 1.

Let a_i be the property of a 5-tuple that its i^{th} digit is an "isolated" 1.

$$N(a_1' a_2' a_3' a_4' a_5') = 2^5 - (2^3 + 2^2 + 2^2 + 2^2 + 2^3) +$$

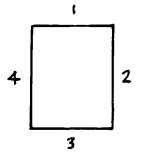
$$(0 + 2 + 1 + 2 + 0 + 1 + 1 + 0 + 2 + 0) - (1) + (0) - (0)$$

$$= 12$$

□

4-5. With three differently colored paints, in how many ways can the walls of a rectangular room be painted so that color changes occur at (and only at) each corner? With two colors?

Define the following properties of a way of painting the walls:



a_{12} - walls 1 and 2 are painted the same color
a_{23} - " 2 " 3 " " " " " "
a_{34} - " 3 " 4 " " " " " "
a_{41} - " 4 " 1 " " " " " "

With three differently colored paints,

$$N(a_{12}' a_{23}' a_{34}' a_{41}') = 3^4 - \binom{4}{1} \cdot 3^3 + \binom{4}{2} \cdot 3^2 - \binom{4}{3} \cdot 3 + \binom{4}{4} \cdot 3$$

$$= 18$$

With two differently colored paints.

$$N(a_{12}' a_{23}' a_{34}' a_{41}') = 2^4 - \binom{4}{1} \cdot 2^3 + \binom{4}{2} \cdot 2^2 - \binom{4}{3} \cdot 2 + \binom{4}{4} \cdot 2$$

$$= 2$$

□

4-6. Among the numbers 1, 2, $\cdots$, 500, how many of them are not divisible by 7 but are divisible by 3 or 5?

Assign properties to the numbers 1, 2, $\cdots$, 500 as follows:

a_3 - divisible by 3
a_5 - " " 5
a_7 - " " 7

$$N(a_3a_5a_7') + N(a_3a_5'a_7') + N(a_3'a_5a_7') = N(a_7') - N(a_3'a_5'a_7')$$

$$N(a_7') = 500 - [\frac{500}{7}] = 429$$

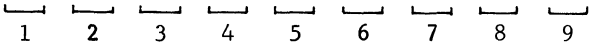
$$N(a_3'a_5'a_7') = 500 - ([\frac{500}{3}] + [\frac{500}{5}] + [\frac{500}{7}]) + ([\frac{500}{3\cdot5}] + [\frac{500}{3\cdot7}] + [\frac{500}{5\cdot7}]) - [\frac{500}{3\cdot5\cdot7}] = 229$$

$$N(a_7') - N(a_3'a_5'a_7') = 200$$

□

4-7. Find the number of permutations of the letters α, α, α, β, β, β, γ, γ, and γ which are such that no identical letters are adjacent.

Consider distributions of the given letters into 9 cells, with one letter per cell. Number the cells as shown below:



Define the following properties of such a distribution:

a_{12} - cells 1 and 2 contain identical letters
a_{23} - " 2 " 3 " " "
$\vdots$
$a_{k,k+1}$ - " k " k+1 " " "
$\vdots$
a_{89} - " 8 " 9 " " "

Then

$$N(a_{12}'a_{23}'\cdots a_{89}') = s_0 - s_1 + s_2 - s_3 + \cdots + s_8$$

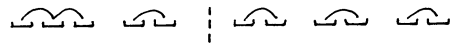
is the desired result.

$$s_0 = \frac{9!}{3!\cdot 3!\cdot 3!} = 1680$$

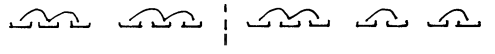
$$s_1 = 8 \cdot (3 \cdot \frac{7!}{3!\cdot 3!}) = 3360$$

$$s_2 = 7 \cdot (3 \cdot \frac{6!}{3!\cdot 3!}) + 21 \cdot (3\cdot 2\cdot \frac{5!}{3!}) = 2940$$

$$s_3 = 30 \cdot (3\cdot 2\cdot \frac{4!}{3!}) + 20 \cdot (3!\cdot 3!) = 1440$$

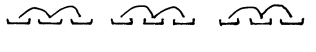


$$s_4 = 10 \cdot (3\cdot 2\cdot \frac{3!}{3!}) + 30 \cdot (3!\cdot 2!) = 420$$



$$s_5 = 12 \cdot (3!\cdot 1!) = 72$$

$$s_6 = 1 \cdot (3!\cdot 0!) = 6$$



$$s_7 = 0$$

$$s_8 = 0$$

$$N(a'_{12}a'_{23}\cdots a'_{89}) = 1680 - 3360 + 2940 - 1440 + 420 - 72 + 6 = 174$$

□

4-8. Find the number of permutations of the 26 letters A, B, ···, X, Y, Z that do not contain the patterns JOHN, PAUL, and SMITH.

Define the following properties of a permutation of the alphabet:

a_1 - contains the string "JOHN"

a_2 - " " " "PAUL"

a_3 - " " " "SMITH"

$$N(a'_1a'_2a'_3) = 26! - (23! + 23! + 22!) + (20! + 0 + 19!) - 0$$

$$= 26! - 2\cdot 23! - 22! + 20! + 19!$$

□

4-9. When two indistinguishable dice are rolled n times:

(a) In how many ways can a pair of each of the six numbers show at least once?

(b) In how many ways can each of the pairs of 3's, 4's, 5's, and 6's show at least once and either a pair of aces or a pair of deuces show at least once?

There are $\binom{6+2-1}{2} = 21$ outcomes for each toss.

Define the following properties of a sequence of n tosses

a_1 - a pair of 1's does not show
a_2 - " " " 2's " " "
⋮
a_6 " " " 6's " " "

(a) $N(a_1'a_2'a_3'a_4'a_5'a_6') = 21^n - \binom{6}{1}\cdot 20^n + \binom{6}{2}\cdot 19^n - \binom{6}{3}\cdot 18^n + \binom{6}{4}\cdot 17^n - \binom{6}{5}\cdot 16^n + \binom{6}{6}\cdot 15^n$

(b) $N(a_3'a_4'a_5'a_6') - N(a_1a_2a_3'a_4'a_5'a_6') = [21^n - \binom{4}{1}\cdot 20^n + \binom{4}{2}\cdot 19^n - \binom{4}{3}\cdot 18^n + \binom{4}{4}\cdot 17^n] -$
$[19^n - \binom{4}{1}\cdot 18^n + \binom{4}{2}\cdot 17^n - \binom{4}{3}\cdot 16^n + \binom{4}{4}\cdot 15^n]$

□

4-10. (a) 5n men, standing in a line, are fron n countries (five men from each country). Show that the number of ways the line can be formed so that every man is standing next to a countryman of his is equal to

$$120^n[(2n)! - \binom{n}{1}(2n-1)! + \binom{n}{2}(2n-2)! - \cdots + (-1)^n\binom{n}{n}n!]$$

(b) Repeat part (a) with 4n men from n countries (four men from each country).

(a) For each country, form an ordered pair and an ordered triple of men. There are $(5!)^n$ ways to do this. For each way, permutations of the 2n pairs and triples give formations in which every man stands next to a countryman. However, to avoid counting the same formation more than once (which occurs for formations in which 5 fellow countrymen are together, e.g., (12)(345) and (123)(45)), define the following properties of an arrangement of a given set of pairs and triples:

a_i - country i triple is immediately to the left of country i pair, for i = 1, 2, ⋯, n.

Then $N(a_1'a_2'\cdots a_n')$ is the number of arrangements in which any country grouped together results from a pair-triple adjacency and not a triple-pair adjacency.

$$N(a_1'a_2'\cdots a_n') = (2n)! - \binom{n}{1}\cdot(2n-1)! + \binom{n}{2}\cdot(2n-2)! - \cdots + (-1)^n\binom{n}{n}\cdot n!$$

and $(5!)^n \cdot N(a_1'a_2'\cdots a_n')$ is the desired result.

(b) There are 12^n ways to form 2n ordered pairs of fellow countrymen, and

$$12^n(2n)!$$

acceptable formations.

□

4-11. The Euler ϕ-function of a positive integer n, $\phi(n)$, is defined as the number of positive integers less than or equal to n that are prime to n. Show that

$$\phi(n) = n \prod_{i=1}^{m} (1 - \frac{1}{p_i})$$

where $p_1, p_2, \cdots, p_m$ are all the distinct prime divisors of n.

Define the following properties of numbers in the set $\{1, 2, \cdots, n\}$:

a_1 - divisable by p_1
a_2 - " " p_2
$\vdots$
a_m - " " p_m

$$\phi(n) = N(a_1'a_2'\cdots a_m')$$

$$= n - (\frac{n}{p_1} + \frac{n}{p_2} + \cdots + \frac{n}{p_m}) + (\frac{n}{p_1p_2} + \frac{n}{p_1p_3} + \cdots + \frac{n}{p_{m-1}p_m})$$

$$- (\frac{n}{p_1p_2p_3} + \frac{n}{p_1p_2p_4} + \cdots + \frac{n}{p_{m-2}p_{m-1}p_m}) + \cdots + (-1)^m \frac{n}{p_1p_2\cdots p_m}$$

$$= n \left[\frac{(p_1-1)(p_2-1)\cdots(p_m-1)}{p_1p_2\cdots p_m}\right]$$

$$= n \prod_{i=1}^{m} (1 - \frac{1}{p_i})$$

□

4-12. Use a combinatorial argument to prove the identity

$$\binom{r-1}{n-m-1} = \sum_{k=0}^{n-m} (-1)^k \binom{n-m}{k}\binom{n-m-k+r-1}{r}$$

Hint: the number of ways to distribute r non-distinct objects into n distinct cells leaving exactly m cells empty is equal to $\binom{n}{m}\binom{r-1}{n-m-1}$.

Use the principle of inclusion and exclusion to count the number of ways to distribute r non-distinct objects into n distinct cells leaving m cells empty.

There are $\binom{n}{m}$ ways to pick m empty cells. For each of these, define the following properties of a way of distributing the r non-distinct objects into the remaining (n-m) distinct cells:

a_1 - the first of these (n-m) cells is empty
a_2 - " second " " " " " "
$\vdots$
a_{n-m} - " $(n-m)^{th}$ " " " " " "

Then,

$$\binom{n}{m}\binom{r-1}{n-m-1} = \binom{n}{m} N(a_1'a_2'\cdots a_{n-m}')$$

$$= \binom{n}{m}\left[\binom{(n-m)+r-1}{r} - \binom{n-m}{1}\binom{(n-m-1)+r-1}{r}\right.$$

$$\left. + \binom{n-m}{2}\binom{(n-m-2)+r-1}{r} - \cdots + (-1)^{n-m}\binom{n-m}{n-m}\binom{r-1}{r}\right]$$

$$= \binom{n}{m}\sum_{k=0}^{n-m}(-1)^k\binom{n-m}{k}\binom{(n-m-k)+r-1}{r}$$

□

4-13. Use a combinatorial argument to prove the identity

$$\binom{n-m}{n-r} = \sum_{k=0}^{m}(-1)^k\binom{m}{k}\binom{n-k}{r} \qquad n \geq r \geq m$$

Hint: $\binom{n-m}{n-r}$ is the number of ways to select r objects from n distinct ones such that m special objects are always included in the selections.

Let a_i, i = 1, 2, $\cdots$, m, be the property of a selection that special object i is not included. Then,

$$\binom{n-m}{n-r} = N(a_1'a_2'\cdots a_m')$$

$$= \binom{n}{r} - \binom{m}{1}\binom{n-1}{r} + \binom{m}{2}\binom{n-2}{r} - \binom{m}{3}\binom{n-3}{r} + \cdots + (-1)^m\binom{m}{m}\binom{n-m}{r}$$

□

4-14. Show that

$$s_m = \sum_{j=m}^{r}\binom{j}{m}e_j$$

where s_m and e_j are defined as in Sec. 4-3. What is the combinatorial significance of this identity?

$$E(x) = \sum_{j=0}^{r} s_j (x-1)^j = \sum_{j=0}^{r} e_j x^j \qquad \text{(Equation 4-7)}$$

$$E'(x) = \sum_{j=1}^{r} (j)\, s_j (x-1)^{j-1} = \sum_{j=1}^{r} (j)\, e_j x^{j-1}$$

$$E''(x) = \sum_{j=2}^{r} (j)(j-1)\, s_j (x-1)^{j-2} = \sum_{j=2}^{r} (j)(j-1)\, e_j x^{j-2}$$

$$\vdots$$

$$E^{(m)}(x) = m! \sum_{j=m}^{r} \binom{j}{m} s_j (x-1)^{j-m} = m! \sum_{j=m}^{r} \binom{j}{m} e_j x^{j-m}$$

$$E^{(m)}(1) = m! \binom{m}{m} s_m = m! \sum_{j=m}^{r} \binom{j}{m} e_j$$

$$s_m = \sum_{j=m}^{r} \binom{j}{m} e_j$$

An object having j properties ($j < m$) contributes nothing to the count s_m.

An object having j properties ($j \geq m$) contributes $\binom{j}{m}$ to the count s_m.

□

4-15. (a) Prove the identity

$$\binom{m+r}{m} - \binom{m+r}{m+1} + \binom{m+r}{m+2} - \cdots + (-1)^r \binom{m+r}{m+r} = \binom{m+r-1}{m-1}$$

(b) Let $e_{\geq m}$ denote the number of objects having m or more of r properties; that is,

$$e_{\geq m} = e_m + e_{m+1} + e_{m+2} + \cdots + e_r$$

Prove that

$$e_{\geq m} = s_m - \binom{m}{m-1} s_{m+1} + \binom{m+1}{m-1} s_{m+2} - \binom{m+2}{m-1} s_{m+3} + \cdots + (-1)^{r-m} \binom{r-1}{m-1} s_r$$

(a) The left hand side can be written as

$$\binom{m+r}{r} - \binom{m+r}{r-1} + \binom{m+r}{r-2} - \cdots + (-1)^r \binom{m+r}{0}$$

which is the coefficient of x^r in

$$(1+x)^{m+r} - x(1+x)^{m+r} + x^2(1+x)^{m+r} - \cdots + (-1)^r x^r (1+x)^{m+r}$$

$$= (1+x)^{m+r} \left(\frac{1 - (-1)^{r+1} x^{r+1}}{1+x}\right) = (1+x)^{m+r-1} \left(1 - (-1)^{r+1} x^{r+1}\right)$$

The coefficient of x^r is $\binom{m+r-1}{r} = \binom{m+r-1}{m-1}$

(b) From Equation 4-6

$$e_m = s_m - \binom{m+1}{m} s_{m+1} + \binom{m+2}{m} s_{m+2} - \cdots + (-1)^{n-m} \binom{n}{m} s_n$$

$$e_{m+1} = s_{m+1} - \binom{m+2}{m+1} s_{m+2} - \cdots + (-1)^{n-m-1} \binom{n}{m+1} s_n$$

$$e_{m+2} = s_{m+2} - \cdots + (-1)^{n-m-2} \binom{n}{m+2} s_n$$

$$\vdots \qquad\qquad \vdots$$

$$e_n = s_n$$

Using part (a), the identity follows.

□

4-16. Two professors in two different subjects are giving oral examinations to 12 students in the same hour. Each student shall be examined individually for 5 minutes in each subject. In how many ways can a schedule be made up so that no student will have to see both professors at the same time? Give an approximated value to the answer.

For each of the 12! exam schedules for one subject, there are d_{12} exam schedules, for the other subject giving

$$12!\, d_{12} \approx \frac{(12!)^2}{e} \quad \text{possible exam schedules.}$$

□

4-17. (The Game of Rencontres) There are n balls, numbered 1, 2, ···, n, in an urn. When the balls are drawn one by one without replacement from the urn, a rencontre is said to occur if the ball numbered r appears in the rth drawing. Find the probability of at least one recontre and the probability of exactly m rencontres. Give approximated values of the probabilities.

Number of sequences with no recontres = d_n

" " " " one or more recontres = $n! - d_n$

Probability of such a sequence = $\frac{n! - d_n}{n!} \approx 1 - e^{-1}$

Number of sequences with exactly m recontres =

Number of sequences in which exactly (n-m) balls are deranged =

$$\binom{n}{m} d_{n-m} \approx \frac{n!}{m!} e^{-1}$$

Probability of such a sequence = $\dfrac{\binom{n}{m} d_{n-m}}{n!} \approx \dfrac{e^{-1}}{m!}$

□

4-18. The n integers 1, 2, ···, n are arranged around a circle. Let g_n denote the number of such arrangements in which no two adjacent integers are consecutive in clockwise order (n and 1 are considered to be consecutive). Find an expression for g_n, and show that it satisfies the recurrence equation

$$g_n + g_{n+1} = d_n \qquad n \geq 1$$

Define properties of a circular arrangement as follows:

a_{12} - contains the ordered pair (1,2) in clockwise order
a_{23} - " " " " (2,3) " " "
⋮
a_{n1} " " " " (n,1) " " "

$g_n = N(a'_{12}a'_{23}\cdots a'_{n1}) =$

$$(n-1)! - \binom{n}{1}(n-2)! + \binom{n}{2}(n-3)! - \binom{n}{3}(n-4)! + \cdots + (-1)^{n-1}\binom{n}{n-1}0! + (-1)^n \binom{n}{n}$$

$$g_n + g_{n+1} = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k}(n-k-1)! + (-1)^n \binom{n}{n} + \sum_{k=0}^{n} (-1)^k \binom{n+1}{k}(n-k)! + (-1)^{n+1}\binom{n+1}{n+1}$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k-1}(n-k)! + n! + \sum_{k=1}^{n} (-1)^k \binom{n+1}{k}(n-k)!$$

$$= n! + \sum_{k=1}^{n} (-1)^k \left[\binom{n+1}{k} - \binom{n}{k-1}\right](n-k)!$$

$$= n! + \sum_{k=1}^{n} (-1)^k \binom{n}{k}(n-k)! = d_n$$

□

4-19. (a) Show that the number of permutations of the n integers 1, 2, ···, n in which no two adjacent integers are consecutive in left-to-right order is equal to $d_n + d_{n-1}$.

(b) If n and 1 are also considered as "consecutive numbers," show that the number of permutations of the n integers 1,2, ···, n in which no two adjacent integers are "consecutive" is equal to nd_{n-1}.

(a) Define properties of a permutation as follows:

a_{12} - contains the ordered pair (1,2)
a_{23} - " " " " (2,3)
$\vdots$
$a_{n-1,n}$ - " " " " (n-1, n)

$$N(a'_{12}a'_{23}\cdots a'_{n-1,n}) = n! - \binom{n-1}{1}(n-1)! + \binom{n-1}{2}(n-2)! - \binom{n-1}{3}(n-3)! + \cdots + (-1)^{n-1}\binom{n-1}{n-1}\cdot 1!$$

$$d_n + d_{n-1} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)! + \sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}(n-k)!$$

$$= n! + \sum_{k=1}^{n}(-1)^k\left[\binom{n}{k} - \binom{n-1}{k-1}\right](n-k)!$$

$$= n! + \sum_{k=1}^{n-1}(-1)^k\binom{n-1}{k}(n-k)!$$

$$= \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}(n-k)! = N(a'_{12}a'_{23}\cdots a'_{n-1,n})$$

(b) Define the additional property $a_{n,1}$ of a permutation containing the ordered pair (n,1).

$$N(a'_{12}a'_{23}\cdots a'_{n-1,n}a'_{n,1}) =$$

$$n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! + \cdots + (-1)^{n-1}\binom{n}{n-1}1! + (-1)^n\binom{n}{n}\cdot 0$$

$$nd_{n-1} = n\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}(n-k-1)!$$

$$= \sum_{k=0}^{n-1}(-1)^k\frac{n}{n-k}\binom{n-1}{k}(n-k)!$$

$$= \sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)! = N(a'_{12}a'_{23}\cdots a'_{n-1,n}a'_{n,1})$$

□

4-20. (The Permanent) Let $A = [a_{ij}]$ be an $m \times n$ matrix with $m \le n$. Let A_r denote a matrix obtained from A by replacing r of the n columns of A by 0's. Let $S(A_r)$ be the product of the row sums of A_r and $\Sigma S(A_r)$ be the sum of the $S(A_r)$'s over all A_r's. Define the permanent of A, denoted by Per (A), as

$$\text{Per } (A) = \Sigma a_{1i_1}a_{2i_2}\cdots a_{mi_m}$$

where the summation extends over all m-permutations $(i_1, i_2, \cdots, i_m)$ of the n integers 1, 2, $\cdots$, n.

(a) Show that

$$\text{Per } (A) = \sum S(A_{n-m}) - \binom{n-m+1}{1} \sum S(A_{n-m+1}) + \binom{n-m+2}{2} \sum S(A_{n-m+2}) - \cdots + (-1)^{m-1} \binom{n-1}{m-1} \sum S(A_{n-1})$$

Hint: Consider as the jth property of a term $a_{1i_1} a_{2i_2} \cdots a_{mi_m}$ that $i_p \neq j$ $(p = 1,2,\cdots,m)$.

(b) Let I be the nXn identity matrix, and let K be the nXn matrix whose entries are all 1's. Show that

1. Per (K) = n!

2. Per (K - I) = d_n, where d_n is the number of derangements of n objects.

(a) Define the following properties of a term $a_{1i_1} a_{2i_2} \cdots a_{mi_m}$, where $(i_1, i_2, \cdots, i_m)$ is an m-tuple from the set $\{1,2,\cdots,n\}$.

a_1 - $i_p \neq 1$, $p = 1,2,\cdots,m$
a_2 - $i_p \neq 2$, $p = 1,2,\cdots,m$
$\vdots$
a_n - $i_p \neq n$, $p = 1,2,\cdots,m$

Per (A) is the sum of all terms having exactly (n-m) of these properties.
$\sum S(A_r)$ generates the sum of all terms having r or more of these properties.

Applying Equation 4-6,

$$\text{Per } (A) = \sum S(A_{n-m}) - \binom{n-m+1}{1} \sum S(A_{n-m+1}) + \binom{n-m+2}{2} \sum S(A_{n-m+2}) - \cdots + (-1)^{m-1} \binom{n-1}{m-1} \sum S(A_{n-1}) + (-1)^m \binom{n}{m} \hat{\sum} S(A_n)$$

(b) $$\text{Per } (K) = \sum S(A_0) - \sum S(A_1) + \sum S(A_2) - \cdots + (-1)^{n-1} \sum S(A_{n-1}) + (-1)^n \hat{\sum} S(A_n)$$

$$= \binom{n}{0} \cdot n^n - \binom{n}{1} \cdot (n-1)^n + \binom{n}{2} \cdot (n-2)^n - \cdots + (-1)^{n-1} \binom{n}{n-1} \cdot 1^n + (-1)^n \binom{n}{n} \cdot 0^n$$

But this is also $N(a_1' a_2' \cdots a_n')$, where a_i, $i = 1,2,\cdots,n$, is the property of a distribution of n distinct objects into n distinct cells that the i^{th} cell is left empty, so that

$$\text{Per } (K) = n!$$

This also follows directly from the definition of Per (K),

$$\text{Per}(K) = \sum k_{1i_1} k_{2i_2} \cdots k_{ni_n} = \underbrace{1 + 1 + \cdots + 1}_{n! \text{ terms}} = n!$$

Let $B = K - I$

$$\text{Per}(B) = \sum b_{1i_1} b_{2i_2} \cdots b_{ni_n}$$

$$b_{1i_1} b_{2i_2} \cdots b_{ni_n} = \begin{cases} 1 & \text{if } i_p \neq p \text{ for } p = 1,2,\cdots,n \\ 0 & \text{otherwise} \end{cases}$$

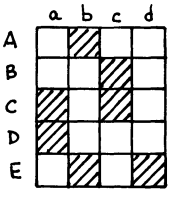
$$\text{Per}(B) = \underbrace{1 + 1 + \cdots + 1}_{d_n \text{ terms}} + \underbrace{0 + 0 + \cdots + 0}_{n! - d_n \text{ terms}} = d_n$$

□

4-21. (a) A computer matching service has five male subscribers A, B, C, D, and E and four female subscribers a, b, c, and d. After analyzing their personalities and interests, the computer decides that a should not be matched with C and D, b should not be matched with A and E, c should not be matched with B and C, and d would quarrel with E. Find the number of ways in which the subscribers can be matched.

(b) Since there are five boys and only four girls, one of the boys will be left with no date. However, B is a good friend of the programmer who programs the computer and sees to it that B is matched. In how many ways can the subscribers be matched?

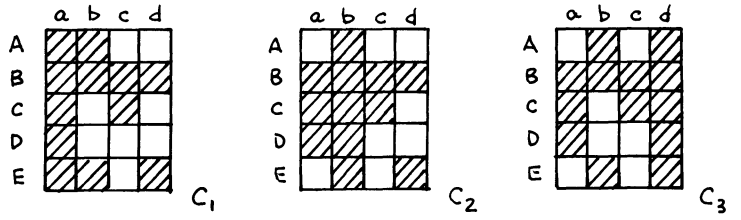
(a) Find r_4 in the rook polynomial for the white chessboard C below.



By repeated application of the expansion theorem (Equation 4-10) it is found that

$$r_4(C) = 25$$

(b) Form the reduced chessboard for each of the three ways B can be matched.



$$r_3(C_1) + r_3(C_2) + r_3(C_3) = 7 + 7 + 6 = 20$$

□

4-22. Find the rook polynomial $R_{m,\ell}(x)$ for the staircase chessboard shown in Fig. 4P-1.

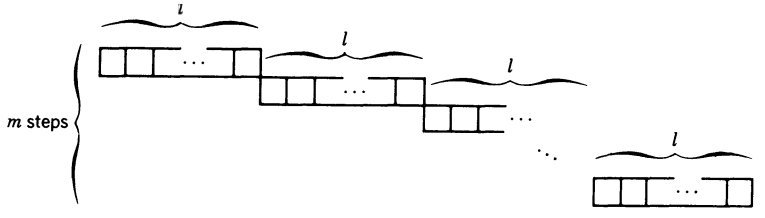


Fig. 4P-1

$$R_{1,\ell}(x) = 1 + \ell x$$

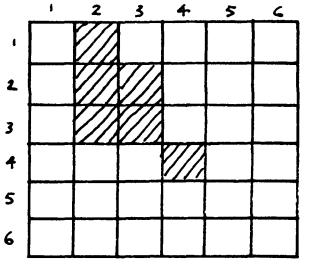
By application of Equation 4-11, since the steps are pairwise disjunct,

$$R_{m,\ell}(x) = (1 + \ell x)^m$$

□

4-23. A pair of distinct dice are rolled six times. It is known that the combinations (1,2), (2,2), (3,2), (2,3), (3,3), and (4,4) did not appear. Find the probability that all faces of the two dice have shown.

The number of ways that all faces of the two dice can show in 6 tosses without respect to order is equal to r_6 for the white chessboard C_1 below.



Further, $r_6(C_1)$ = number of permutations of $\{1,2,3,4,5,6\}$ with forbidden positions corresponding to the darkened chessboard C_2 above.

Let a_i, $i = 1,2,\cdots,6$, be the property of a permutation in which the i^{th} forbidden position is occupied. Then

$$r_6(C_1) = N(a_1'a_2'\cdots a_6') = \sum_{j=0}^{6} (-1)^j \, r_j(C_2)\cdot(6-j)!$$

Using the expansion theorem

$$R(x,C_2) = 1 + 6x + 9x^2 + 4x^3$$

and

$$r_6(C_1) = 1\cdot 6! - 6\cdot 5! + 9\cdot 4! - 4\cdot 3! = 192$$

Each of these 192 ways is an unordered collection of 6 distinct ordered pairs, and can occur in 6! ways. Thus there are $192\cdot 6!$ favorable events. Since there are 30^6 possible events, the desired probability is

$$\frac{192\cdot 6!}{30^6} = \frac{2^4}{3^3 5^5}$$

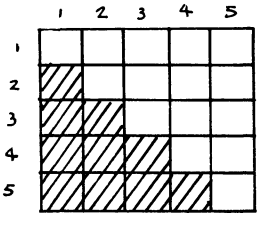
□

4-24. An arrangement of integers is said to contain a descent when an integer is immediately followed by a smaller integer in the arrangement. For example, the arrangement 52143 contains three descents.

(a) Find the chessboard for the restrictions in the enumeration of the number of arrangements of the integers 1, 2, 3, 4, and 5 that contain no descent. The restriction implied by the cell (i,j) in this chessboard is now interpreted as the integer i being to the immediate left of the integer j in an arrangement, for $i \neq j$.

(b) Find the rook polynomial of the chessboard in part (a). Show that its corresponding hit polynomial is the ordinary generating function that enumerates the number of arrangements containing zero, one, two, three, and four descents, respectively.

(a) Restrictions are specified by the darkened chessboard C.



(b) Using the expansion theorem

$$R(x,C) = 1 + 10x + 25x^2 + 15x^3 + x^4$$

The corresponding hit polynomial is

$$E(x) = \sum_{j=0}^{5} r_j(C)\cdot(5-j)!\ (x-1)^j$$

$$= 1\cdot 5! + 10\cdot 4!\ (x-1) + 25\cdot 3!\ (x-1)^2 + 15\cdot 2!\ (x-1)^3 + 1\cdot 1!\ (x-1)^4$$

$$= 1 + 26x + 66x^2 + 26x^3 + x^4$$

Define properties of a permutation, one for each possible descent. There are $r_j(C)$ possible sets of j descents that can be found in a permutation, and (5-j)! permutations containing each possible set of j descents. Therefore, in Equation 4-7,

$$s_j = r_j(C)\cdot(5-j)!$$

and E(x), as found above, enumerates the permutations having exactly 0,1,2,3,4 descents.

□

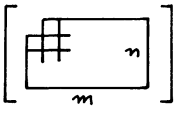
4-25. (a) Use the expansion formula to show that the rook polynomial of an n×m rectangular chessboard, $R_{n,m}(x)$, satisfies the recurrence relation

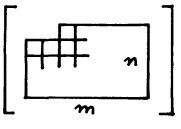
$$R_{n,m}(x) = R_{n-1,m}(x) + mxR_{n-1,m-1}(x)$$

(b) Show that

$$\frac{d}{dx} R_{n,m}(x) = nmR_{n-1,m-1}(x)$$

(c) If the cells in an n×m rectangular chessboard are forbidden positions for the permutations of n elements ($n \geq m$), find the corresponding hit polynomial E(x).

(a) $R_{n,m}(x) = xR_{n-1,m-1}(x) +$ 

$= 2xR_{n-1,m-1}(x) +$ 

$\vdots$

$= mxR_{n-1,m-1}(x) + R_{n-1,m}(x)$

Alternatively,

$$R_{n,m}(x) = nxR_{n-1,m-1}(x) + R_{n,m-1}(x)$$

(b) Let $R_{n,m}(x) = \sum_{k=0}^{\infty} (r_k)_{n,m} x^k$

Compute $(r_k)_{n,m}$ in terms of $(r_{k-1})_{n-1,m-1}$.

For an initial placement of a rook on any one of the nm squares, there are $(r_{k-1})_{n-1,m-1}$ ways to place (k-1) remaining non-taking rooks. Summing over all possible initial placements, gives a count of $nm(r_{k-1})_{n-1,m-1}$. However, any particular placement of k non-taking rooks on the nxm chessboard is counted exactly k times in this manner, once each time an occupied square is selected for the intial placement.

Therefore,

$$k(r_k)_{n,m} = nm(r_{k-1})_{n-1,m-1} \qquad [1]$$

Multiplying by x^{k-1} and summing over $k = 1,2,\cdots$

$$\sum_{k=1}^{\infty} k(r_k)_{n,m} x^{k-1} = nm \sum_{k=1}^{\infty} (r_{k-1})_{n-1,m-1} x^{k-1}$$

or

$$\frac{d}{dx} R_{n,m}(x) = nmR_{n-1,m-1}(x)$$

Alternatively, the recurrence relation [1] can be established by noting that $(r_k)_{n,m}$ has a closed form value given by,

$$(r_k)_{n,m} = \binom{n}{k} P(m,k) = \binom{m}{k} P(n,k)$$

and that,

$$k\left[\frac{n!}{(n-k)!\ k!} \cdot \frac{m!}{(m-k)!}\right] = nm\left[\frac{(n-1)!}{(n-k)!(k-1)!} \cdot \frac{(m-1)!}{(m-k)!}\right]$$

(c) $R_{n,m}(x) = 1 + \binom{n}{1} P(n,1)x + \binom{n}{2} P(n,2)x^2 + \cdots + \binom{n}{m} P(n,m)x^m$

$$E(x) = \sum_{k=0}^{m} (r_k)_{n,m} (n-k)! (x-1)^k$$

$$= \sum_{k=0}^{m} \left[\frac{n!}{(n-k)!\ k!} \cdot \frac{m!}{(m-k)!}\right] (n-k)! (x-1)^k$$

$$= n! \sum_{k=0}^{m} \binom{m}{k} (x-1)^k$$

$$= n! [1 + (x-1)]^m$$

$$= n! x^m$$

This should be expected, since every permutation occupies exactly m forbidden positions.

□

4-26. (Equivalence) (a) Let C_1 be a chessboard that has l columns and at the most n rows. Let C be an nXm rectangular chessboard. The chessboard $C_1 + C$ is defined as a board containing all the cells in C_1 and C arranged as in Fig. 4P-2. Let

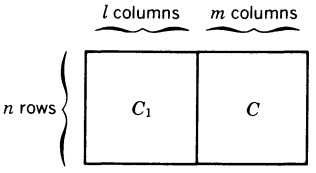


Figure 4P-2

$R(x,C_1) = \sum_k r_k x^k$, and let $R(x, C_1 + C) = \sum_k q_k x^k$. Find q_k, and express it in terms of the r_k's.

(b) Two chessboards C_1 and C_2 are said to be equivalent if $R(x,C_1) = R(x,C_2)$. Show that if C_1 and C_2 are equivalent, $C_1 + C$ and $C_2 + C$ are also equivalent.

(a) $$q_k = r_0\binom{m}{k} P(n,k) + r_1\binom{m-1}{k-1} P(n,k-1) + r_2\binom{m-2}{k-2} P(n,k-2) + \cdots + r_k\binom{m-k}{0} P(n,0)$$

(b) $$R(x,C_1) = \sum_j r_j x^j = R(x,C_2)$$

$$R(x,C_1 + C) = \sum_k q_k x^k = \sum_k \Big(\sum_{j=0}^{k} r_j\binom{m-j}{k-j} P(n,k-j)\Big)x^k = R(x,C_2 + C)$$

□

4-27. (The Complement of a Chessboard) The complement C' of a chessboard C with respect to an nXm rectangular chessboard $C_{n,m}$ is defined as the chessboard that contains all the cells that are in $C_{n,m}$ but not in C. (Assume that C has at the most n rows and m columns).

(a) Let r_k and q_k be the number of ways to place k non-taking rooks on a chessboard C and its complement C' with respect to $C_{n,m}$, respectively. Show that

$$q_k = \binom{n}{k} P(m,k) - r_1\binom{n-1}{k-1} P(m-1,k-1) + r_2\binom{n-2}{k-2} P(m-2,k-2) - \cdots + (-1)^k r_k \binom{n-k}{0} P(m-k,0)$$

(b) Let $R_{i,j}(x)$ denote the rook polynomial of an ixj rectangular chessboard. Show that the rook polynomial of C', Q(x), can be expressed as

$$Q(x) = \sum_{k=0}^{\min[n,m]} q_k x^k = \sum_{k=0}^{\min[n,m]} (-1)^k r_k x^k R_{n-k,m-k}(x)$$

(c) Let C and C' be complementary chessboards with respect to an m×n chessboard, and let E(x,C) and E(x,C') be their hit polynomials. Show that

$$E(x,C) = x^n E(\frac{1}{x}, C')$$

$$E(x,C') = x^n E(\frac{1}{x}, C)$$

(a) Define properties of a placement of k non-taking rooks on $C_{n,m}$

a_1 - the $1^{\underline{st}}$ cell of C is occupied

a_2 - " $2^{\underline{nd}}$ " " " " "

$\vdots$

a_{r_1} - " $r_1^{\underline{th}}$ " " " " "

$q_k = N(a_1' a_2' \cdots a_{r_1}')$

The given equation follows directly.

(b)
$$Q(x) = \sum_{k=0}^{\min[n,m]} q_k x^k = \sum_{k=0}^{\min[n,m]} \left(\sum_{j=0}^{k} (-1)^j r_j \binom{n-j}{k-j} P(m-j,k-j) \right) x^k$$

$$= \sum_{j=0}^{\min[n,m]} (-1)^j r_j \sum_{k=j}^{\min[n,m]} \binom{n-j}{k-j} P(m-j, k-j) x^k$$

$$= \sum_{j=0}^{\min[n,m]} (-1)^j r_j x^j \sum_{k=j}^{\min[n,m]} \binom{n-j}{k-j} P(m-j,k-j) x^{k-j}$$

$$= \sum_{j=0}^{\min[n,m]} (-1)^j r_j x^j \; R_{n-j,m-j}(x)$$

(c) Let $E(x,C) = \sum_{k=0}^{n} e(C)_k x^k$

$$E(x,C') = \sum_{k=0}^{n} e(C')_k x^k$$

Because C and C' are complementary chessboards with respect to $C_{n,n}$,

$$e(C)_k = e(C')_{n-k}$$

Then,

$$E(x,C) = \sum_{k=0}^{n} e(C')_{n-k} x^k$$

$$= x^n \sum_{k=0}^{n} e(C')_{n-k} \frac{1}{x^{n-k}}$$

$$= x^n E(\frac{1}{x}, C')$$

Similarly,

$$E(x,C') = \sum_{k=0}^{n} e(C)_{n-k} x^k$$

$$= x^n \sum_{k=0}^{n} e(C)_{n-k} \frac{1}{x^{n-k}}$$

$$= x^n E(\frac{1}{x}, C)$$

□

4-28. (Triangular Chessboard) Let $T_n(x)$ be the rook polynomial of an n×n triangular chessboard as shown in Fig. 4P-3. We shall prove by mathematical induction that

$$T_n(x) = \sum_{k=0}^{n} S(n+1,n+1-k)x^k \qquad (4P\text{-}1)$$

where S(i,j) is the Stirling number of the second kind.

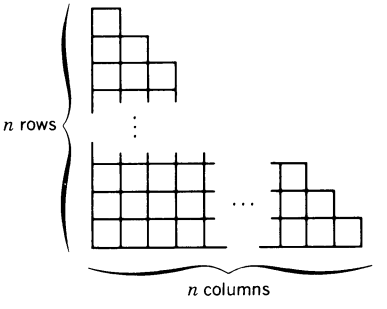


Fig. 4P-3

(a) Basis of induction: Show that Eq. (4P-1) is satisfied for n = 1 and n = 2.

(b) Induction step: Assume that $T_m(x) = \sum_{k=0}^{m} S(m+1,m+1-k)x^k$ for $m \leq n - 1$. Show that Eq. (4P-1) is true.

Hint: $S(n+1,k) = \sum_{j=0}^{n} \binom{n}{j} S(j,k-1)$

(a) Basis. $n = 1$ $T_1(x) = S(2,2) + S(2,1)x = 1 + x$

$T_2(x) = S(3,3) + S(3,2)x + S(3,1)x^2 = 1 + 3x + x^2$

(b) Induction Step.

For an n×n triangular chessboard, the reduced chessboard corresponding to inclusion of any k main diagonal cells, $k = 0,1,\cdots,n-1$, is an (n-k) x(n-k) triangular chessboard. The reduced chessboard corresponding to inclusion of exactly k main diagonal cells, $k = 0,1,\cdots,n-1$, (the remaining (n-k) main diagonal cells are discarded) is an (n-k-1) x (n-k-1) triangular chessboard.

Expanding with respect to the n main diagonal cells,

$$T_n(x) = \binom{n}{0} T_{n-1}(x) + \binom{n}{1} xT_{n-2}(x) + \binom{n}{2} x^2T_{n-3}(x) + \cdots + \binom{n}{n-2} x^{n-2}T_1(x) + \binom{n}{n-1} x^{n-1}T_0(x) + \binom{n}{n} x^n$$

$$= \sum_{j=0}^{n-1} \binom{n}{j} x^j T_{n-j-1}(x) + \binom{n}{n}x^n$$

By induction hypothesis, this can be written as

$$T_n(x) = \sum_{j=0}^{n-1} \binom{n}{j}x^j \sum_{k=0}^{n-j-1} S(n-j,n-j-k)x^k + \binom{n}{n}x^n$$

$$= \sum_{j=0}^{n-1} \binom{n}{j}x^j \sum_{k=j}^{n-1} S(n-j,n-k)x^{k-j} + \binom{n}{n}x^n$$

$$= \sum_{k=0}^{n-1} x^k \sum_{j=0}^{k} \binom{n}{j} S(n-j,n-k) + \binom{n}{n}x^n$$

$$= \sum_{k=0}^{n-1} x^k \sum_{j=n-k}^{n} \binom{n}{n-j} S(j,n-k) + \binom{n}{n}x^n$$

$$= \sum_{k=0}^{n-1} x^k \sum_{j=0}^{n} \binom{n}{j} S(j,n-k) + \binom{n}{n}x^n$$

Using the following identity (for which a combinatorial argument is given below).

$$\sum_{j=0}^{n} \binom{n}{j} S(j,n-k) = S(n+1,n+1-k) \qquad [1]$$

gives

$$T_n(x) = \sum_{k=0}^{n-1} x^k S(n+1,n+1-k) + \binom{n}{n}x^n$$

$$= \sum_{k=0}^{n} S(n+1,n+1-k)x^k$$

The argument for [1] is as follows. The right hand side is the number of ways of placing (n+1) distinct objects into (n+1-k) non-distinct cells, with no cell left empty. (See Section 2-4). Identify one of the (n+1) distinct objects as a special object. Suppose it is placed in a cell along with (n-j) other objects, j = 0,1,2,$\cdots$,n. There are $\binom{n}{j}$ ways to select the j objects not sharing a cell with the special object, and S(j,n-k) ways to place them in the remaining (n-k) cells, leaving no cell empty. Summing the product $\binom{n}{j}\cdot S(j,n-k)$ over j establishes the identity.

□

4-29. (The Menage Problem) In the problem of enumerating the permutations of the integers 1, 2, $\cdots$, n which are such that the integer i is not in positions i and i + 1 for i = 1, 2, $\cdots$, n-1 and the integer n is not in positions n and 1, the chessboard for the forbidden positions is shown in Fig. 4P-4. The rook polynomial of this chessboard is called the Menage polynomial, which is denoted by $M_n(x)$.

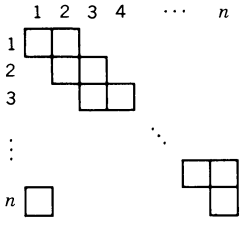


Figure 4P-4

(a) Let $A_{2n-1}(x)$ be the rook polynomial of a staircase chessboard containing an odd number of cells, 2n-1, as shown in Fig. 4P-5a. Let $B_{2n}(x)$ be the rook polynomial of a staircase chessboard containing an even number of cells, 2n, as shown in Fig. 4P-5b. Show that

$$A_{2n-1}(x) = xA_{2n-3}(x) + B_{2n-2}(x)$$

$$B_{2n}(x) = xB_{2n-2}(x) + A_{2n-1}(x)$$

(b) With the boundary conditions

$A_1(x) = 1 + x$ $\qquad$ $A_3(x) = 1 + 3x + x^2$

$B_0(x) = 1$ $\qquad$ $B_2(x) = 1 + 2x$

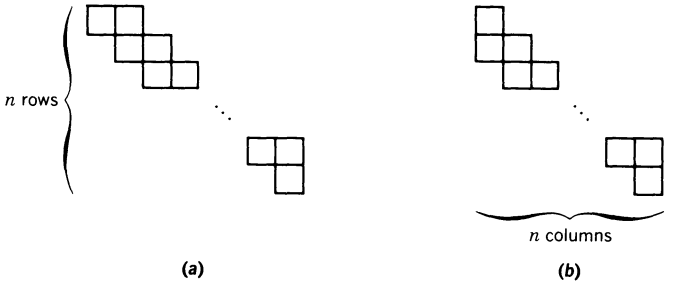


Figure 4P-5

show that

$$A_{2n-1}(x) = \sum_{k=0}^{n} \binom{2n-k}{k} x^k$$

$$B_{2n}(x) = \sum_{k=0}^{n} \binom{2n-k+1}{k} x^k$$

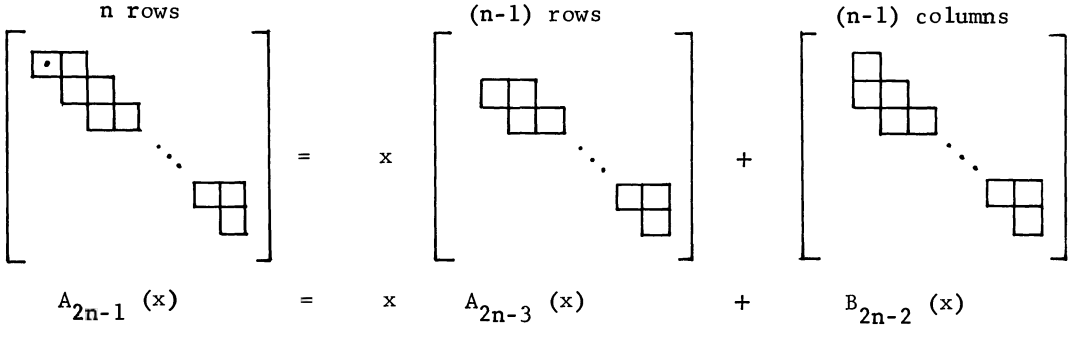
by mathematical induction.

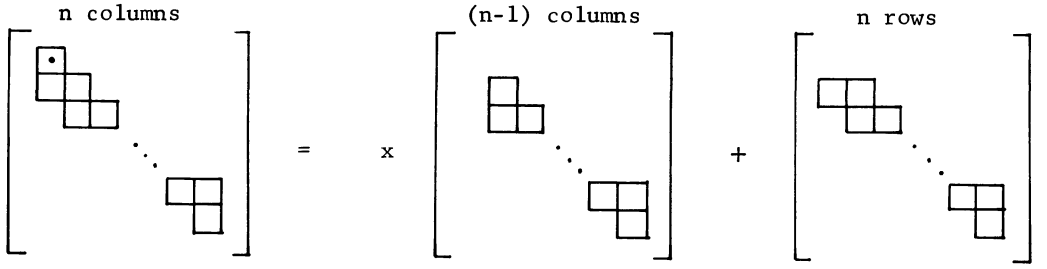
(c) Show that

$$M_n(x) = xA_{2n-3}(x) + A_{2n-1}(x)$$

(d) Find $M_n(x)$ and the corresponding hit polynomial.

(a)





$B_{2n}(x) = x\,B_{2n-2}(x) + A_{2n-1}(x)$

(b) Basis. $A_1(x) = \sum_{k=0}^{1} \binom{2-k}{k} x^k = 1 + x$

$A_3(x) = \sum_{k=0}^{2} \binom{4-k}{k} x^k = 1 + 3x + x^2$

$B_0(x) = \binom{1}{0} = 1$

$B_2(x) = \sum_{k=0}^{1} \binom{3-k}{k} x^k = 1 + 2x$

Induction Hypothesis.

Assume $A_{2n-3}(x) = \sum_{k=0}^{n-1} \binom{2n-2-k}{k} x^k$

$B_{2n-2}(x) = \sum_{k=0}^{n-1} \binom{2n-1-k}{k} x^k$

Induction Step.

Using the recurrence relations of part (a),

$$A_{2n-1}(x) = \sum_{k=0}^{n-1} \binom{2n-2-k}{k} x^{k+1} + \sum_{k=0}^{n-1} \binom{2n-1-k}{k} x^k$$

$$= \sum_{k=1}^{n} \binom{2n-1-k}{k-1} x^k + \sum_{k=0}^{n-1} \binom{2n-1-k}{k} x^k$$

$$= 1 + \sum_{k=1}^{n-1} \left[\binom{2n-1-k}{k-1} + \binom{2n-1-k}{k}\right] x^k + x^n$$

$$= 1 + \sum_{k=1}^{n-1} \binom{2n-k}{k} x^k + x^n$$

$$= \sum_{k=0}^{n} \binom{2n-k}{k} x^k$$

$$B_{2n}(x) = \sum_{k=0}^{n-1} \binom{2n-1-k}{k} x^{k+1} + \sum_{k=0}^{n} \binom{2n-k}{k} x^k$$

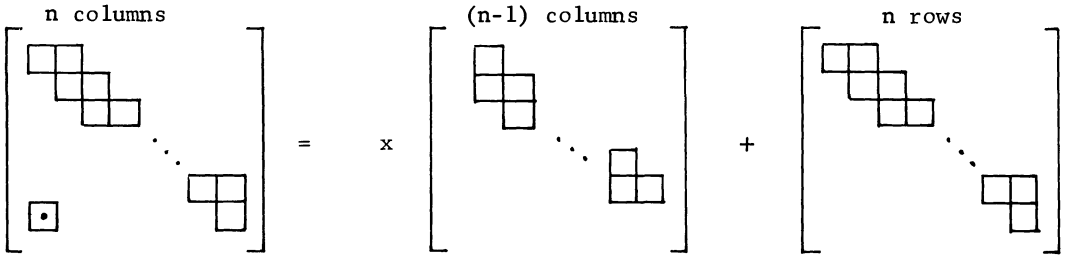
$$= \sum_{k=1}^{n} \binom{2n-k}{k-1} x^k + \sum_{k=0}^{n} \binom{2n-k}{k} x^k$$

$$= 1 + \sum_{k=1}^{n} \left[\binom{2n-k}{k-1} + \binom{2n-k}{k} \right] x^k$$

$$= 1 + \sum_{k=1}^{n} \binom{2n-k+1}{k} x^k$$

$$= \sum_{k=0}^{n} \binom{2n-k+1}{k} x^k$$

(c)



$M_n(x) = x\,A_{2n-3}(x) + A_{2n-1}(x)$

(d) Using the results of parts (b) and (c),

$$M_n(x) = \sum_{k=0}^{n-1} \binom{2n-2-k}{k} x^{k+1} + \sum_{k=0}^{n} \binom{2n-k}{k} x^k$$

$$= \sum_{k=1}^{n} \binom{2n-1-k}{k-1} x^k + \sum_{k=0}^{n} \binom{2n-k}{k} x^k$$

$$= 1 + \sum_{k=1}^{n} \left[\binom{2n-1-k}{k-1} + \binom{2n-k}{k} \right] x^k$$

$$= 1 + \sum_{k=1}^{n} \frac{2n}{2n-k} \binom{2n-k}{k} x^k$$

$$= \sum_{k=0}^{n} \frac{2n}{2n-k} \binom{2n-k}{k} x^k$$

$$E_n(x) = \sum_{k=0}^{n} \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)! \, (x-1)^k$$

□

5-1. Let S be a set containing n elements.

(a) How many different binary relations on the set S are there?

(b) How many of them are reflexive?

(c) How many of them are symmetric?

(d) How many of them are neither reflexive nor symmetric?

(e) How many of them are equivalence relations? (This may be expressed in terms of Stirling numbers of the second kind.)

Let $S = \{a_1, a_2, \cdots, a_n\}$.

(a) Select any number of the n^2 ordered pairs in $S \times S$. There are

$$\sum_{k=0}^{n^2} \binom{n^2}{k} = 2^{n^2}$$ binary relations.

(b) Select all n ordered pairs (a_1,a_1), (a_2,a_2), $\cdots$, (a_n,a_n), and any number of the remaining n^2-n ordered pairs. There are

$$\sum_{k=0}^{n(n-1)} \binom{n(n-1)}{k} = 2^{n(n-1)}$$ reflexive binary relations.

(c) Select any number of the n ordered pairs (a_1,a_1), (a_2,a_2), $\cdots$, (a_n,a_n), and select from the remaining n^2-n ordered pairs in groups of two, i.e., (a_i,a_j) and (a_j,a_i). There are

$$\sum_{k=0}^{n} \binom{n}{k} \cdot \sum_{k=0}^{\frac{1}{2}(n^2-n)} \binom{\frac{1}{2}(n^2-n)}{k} = 2^n \cdot 2^{\frac{1}{2}(n^2-n)}$$

$$= 2^{\frac{n}{2}(n+1)}$$ symmetric binary relations.

(d) Let r denote the reflexive property

and s " " symmetric "

Then

$$N(r's') = N - N(r) - N(s) + N(rs)$$

To determine N(rs), select all n ordered pairs (a_1,a_1), (a_2,a_2), $\cdots$, (a_n,a_n), and select from the remaining n^2-n ordered pairs in groups of two, as in part (c).

$$N(rs) = \sum_{k=0}^{\frac{1}{2}(n^2-n)} \binom{\frac{1}{2}(n^2-n)}{k} = 2^{\frac{n}{2}(n-1)}$$

Then, using the results of parts (a), (b), (c)

$$N(r's') = 2^{n^2} - 2^{n(n-1)} - 2^{\frac{n}{2}(n+1)} + 2^{\frac{n}{2}(n-1)}$$

(e) Count the number of partitions of S. A partition of S into k blocks may be viewed as a distribution of n distinct objects into k non-distinct cells, with no cell left empty. There are S(n,k) ways to do this, where S(n,k) is a Stirling number of the second kind. Summing as k ranges from 1 to n gives

$$\sum_{k=1}^{n} S(n,k)$$

as the number of partitions of S, and thus the number of equivalence relations on S.

□

5-2. Let p_k denote the number of partitions of a set containing k elements, with $p_0 = 1$. Show that

$$p_{n+1} = \sum_{i=0}^{n} \binom{n}{i} p_i$$

From the (n+1) elements, single out a special element a. Divide the partitions of the set into (n+1) classes, each class being characterized by the size of the block containing a. The class of partitions in which a is a block of size n+1-i, i = 0,1,2,$\cdots$,n, contains

$$\binom{n}{n-i} p_i$$

partitions. Summing over all classes,

$$p_{n+1} = \sum_{i=0}^{n} \binom{n}{n-i} p_i$$

$$= \sum_{i=0}^{n} \binom{n}{i} p_i$$

□

For Problems 5-3, 5-4, and 5-5:

Let D be the set of the six faces of the cube.

Let G be a permutation group on D containing permutations corresponding to all possible rotations of the cube. G contains the following permutations on D with cycle structures as given:

(i) The identity permutation - x_1^6

(ii) 3 permutations corresponding to 180° rotations about lines connecting the centers of opposite faces - $x_1^2 x_2^2$

(iii) 6 permutations corresponding to $\pm 90^{\circ}$ rotations about lines connecting the centers of opposite faces - $x_1^2 x_4$

(iv) 6 permutations corresponding to 180° rotations about lines connecting the mid-points of opposite edges - x_2^3

(v) 8 permutations corresponding to $\pm 120^{\circ}$ rotations about lines connecting opposite vertices - x_3^2

$$P_G = \frac{1}{24}(x_1^6 + 3x_1^2x_2^2 + 6x_1^2x_4 + 6x_2^3 + 8x_3^2)$$

□

5-3. The six faces of a cube are to be painted with six different colors, each face with a distinct color. In how many ways can this be done?

If rotations of the cube are not permitted, there are 6! ways to paint the faces, each with a distinct color. Let S be the set of these 6! ways.

Let G' be a permutation group on S containing one permutation π' for each $\pi \in G$. π' permutes each element of S into that element of S which results by applying the rotation of the cube corresponding to π to it.

Applying Burnside's Theorem, the number of equivalence classes in S induced by G' is

$$\frac{1}{|G'|} \sum_{\pi' \in G'} \Psi(\pi')$$

with,

$|G'| = 24$

$\Psi(\pi'_0) = 6!$ where π'_0 is the identity permutation

$\Psi(\pi') = 0$ for all other $\pi' \in G'$

this gives

$$\frac{1}{24} \cdot 6! = 30$$

□

5-4. (a) The six faces of a cube are to be painted with one or more of six different colors. In how many ways can this be done?

(b) One of these six colors is red. Find the number of ways of painting the cube in which exactly three of the faces are painted red.

Let R be the set of six distinct colors.

(a) Find the number of equivalence classes of functions from D to R. Applying Corollary 5-1, this is

$$P_G(6,6,6,6) = \frac{1}{24}(6^6 + 3\cdot 6^2\cdot 6^2 + 6\cdot 6^2\cdot 6 + 6\cdot 6^3 + 8\cdot 6^2) = 2226$$

(b) Let the weight of the color red in R be x, and the weights of all other colors in R be 1.

Find the coefficient of x^3 in the inventory of equivalence classes of functions from D to R. Using Theorem 5-5, this inventory is

$$P_G\Big(\sum_{r\in R} w(r), \sum_{r\in R} w^2(r), \sum_{r\in R} w^3(r), \sum_{r\in R} w^4(r)\Big) = P_G(5+x,\ 5+x^2,\ 5+x^3,\ 5+x^4)$$

$$= \frac{1}{24}\left[(5+x)^6 + 3(5+x)^2\,(5+x^2)^2 + 6(5+x)^2\,(5+x^4) + 6(5+x^2)^3 + 8(5+x^3)^2\right]$$

The coefficient of x^3 is

$$\frac{1}{24}\left[\binom{6}{3}\cdot 5^3 + 3\cdot 10\cdot 10 + 0 + 0 + 8\cdot 10\right] = 120$$

□

5-5. The six faces of a cube are to be painted with four different colors A, B, C, and D. In how many ways can the cube be painted such that two of the faces are painted with color A, two of the faces are painted with color B, one of the faces is painted with color C, and one of the faces is painted with color D?

Let R = {A, B, C, D} and

w(A) = A
w(B) = B
w(C) = C
w(D) = D

Find the coefficient of $A^2 B^2 C D$ in the pattern inventory,

$$P_G(A+B+C+D,\ A^2+B^2+C^2+D^2,\ A^3+B^3+C^3+D^3,\ A^4+B^4+C^4+D^4) =$$

$$\frac{1}{24}\left[(A+B+C+D)^6 + 3(A+B+C+D)^2\,(A^2+B^2+C^2+D^2)^2 + 6(A+B+C+D)^2\,(A^4+B^4+C^4+D^4) + 6(A^2+B^2+C^2+D^2)^3 + 8(A^3+B^3+C^3+D^3)^2\right]$$

The coefficient of A^2B^2CD is

$$\frac{1}{24}\left[\binom{6}{2}\binom{4}{2}\binom{2}{1} + 3\binom{2}{1}\binom{2}{1} + 0 + 0 + 0\right] = 8$$

□

5-6. A rod divided into six segments is to be colored with one or more of n different colors. In how many ways can this be done?

$D = \{1,2,3,4,5,6\}$, the set of segments.

$R = \{c_1, c_2, \cdots, c_n\}$, the set of colors.

$G = \{\binom{123456}{123456}, \binom{123456}{654321}\}$

$$P_G = \frac{1}{2}(x_1^6 + x_2^3)$$

The number of equivalence classes of functions from D to R is

$$P_G(n,n) = \frac{1}{2}(n^6 + n^3) = \frac{n^3}{2}(n^3 + 1)$$

□

5-7. Find the number of 2×4 red and white patterns made up with three red squares and five white squares.

Let D be the set of 8 squares of the 2×4 array.

Let $R = \{r,w\}$

Let G be the permutation group on D containing the identity permutation and the permutation corresponding to a 180° rotation of the 2×4 array.

$$P_G = \frac{1}{2}(x_1^8 + x_2^4)$$

Let $w(r) = r$

$w(w) = w$

Find the coefficient of r^3w^5 in the pattern inventory,

$$P_G(r+w, r^2+w^2) = \frac{1}{2}[(r+w)^8 + (r^2+w^2)^4]$$

The coefficient of r^3w^5 is

$$\frac{1}{2}\binom{8}{3} = 28$$

□

5-8. (a) In how many distinct ways can the sectors of the circle in Fig. 5P-1 be painted with three colors?

(b) Repeat part (a) with the four radii extensions outside the circle removed.

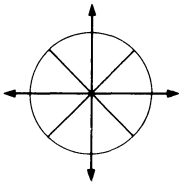


Figure 5P-1

Let D be the set of 8 sectors.

Let R be the set of 3 colors.

(a) Let $G = \{\pi_0, \pi_1, \pi_2, \pi_3\}$ be a permutation group on D, where π_i, i = 0,1,2,3, corresponds to a clockwise rotation of the figure by $i \cdot 90^o$.

Cycle structures are

π_0	x_1^8
π_2	x_2^4
π_1, π_3	x_4^2

$$P_G = \frac{1}{4}(x_1^8 + x_2^4 + 2x_4^2)$$

Using Corollary 5-1, the number of equivalence classes of functions from D to R, with G acting on D, is

$$P_G(3,3,3,3) = \frac{1}{4}(3^8 + 3^4 + 2 \cdot 3^2) = 1665$$

(b) Let $G = \{\pi_0, \pi_1, \pi_2, \cdots, \pi_7\}$ be a permutation group on D, where π_i, i = 0,1,2,$\cdots$,7, corresponds to a clockwise rotation of the figure by $i \cdot 45^o$.

Cycle structures are

π_0	x_1^8
π_4	x_2^4
π_2, π_6	x_4^2
$\pi_1, \pi_3, \pi_5, \pi_7$	x_8

$$P_G = \frac{1}{8}(x_1^8 + x_2^4 + 2x_4^2 + 4x_8)$$

$$P_G(3,3,3,3) = \frac{1}{8}(3^8 + 3^4 + 2\cdot 3^2 + 4\cdot 3) = 834$$

□

5-9. Find the number of distinct ways of painting five of the eight vertices of a cube black and painting the remaining three white.

Let D be the set of 8 vertices.

Let R = {b,w}.

Let G be the permutation group on D corresponding to all possible rotations of the cube.

From Example 5-9,

$$P_G = \frac{1}{24}(x_1^8 + 9x_2^4 + 6x_4^2 + 8x_1^2x_3^2)$$

Let w(b) = b

w(w) = w

The pattern inventory is

$$P_G(b+w, b^2+w^2, b^3+w^3, b^4+w^4) = \frac{1}{24}[(b+w)^8 + 9(b^2+w^2)^4 + 6(b^4+w^4)^2 + 8(b+w)^2(b^3+w^3)^2]$$

The coefficient of b^5w^3 is

$$\frac{1}{24}[\binom{8}{3} + 0 + 0 + 8\cdot\binom{2}{1}] = 3$$

□

5-10. Show that $n^8 + 17n^4 + 6n^2$ is divisible by 24 for any positive integer n.

Hint: Consider the distinct ways of painting the vertices of a cube with n colors.

From Example 5-9, the cycle index for the permutation group G on the vertices of a cube corresponding to all possible rotations of the cube is

$$P_G = \frac{1}{24}[x_1^8 + 9x_2^4 + 6x_4^2 + 8x_1^2x_3^2]$$

The number of ways of painting the vertices with one or more of n colors is

$$P_G(n,n,n,n) = \frac{1}{24}(n^8 + 9n^4 + 6n^2 + 8n^4)$$

$$= \frac{1}{24}(n^8 + 17n^4 + 6n^2)$$

and this must be integral.

□

5-11. Let D be the set of the four faces of a regular pyramid, and let G be the group of all permutations of D that can be produced by the rotation of the pyramid.

(a) Find the cycle index of G.

(b) One or more of four colors, gold, red, white, and blue, are used to paint the faces of the pyramid. Find the number of distinct colorings.

(a) G contains the following permutations on D, with cycle structures as given.

(i) The identity permutation - x_1^4

(ii) 8 permutations corresponding to $\pm 120^\circ$ rotations about lines connecting a vertex and the center of the opposite face - x_1x_3

(iii) 3 permutations corresponding to 180° rotations about lines connecting midpoints of opposite edges - x_2^2

$$P_G = \frac{1}{12}(x_1^4 + 8x_1x_3 + 3x_2^2)$$

(b) R = {gold, red, white, blue}

The number of equivalence classes of functions from D to R is

$$P_G(4,4,4) = \frac{1}{12}(4^4 + 8\cdot4\cdot4 + 3\cdot4^2) = 36$$

□

5-12. Find the number of ways to distribute six balls, three red, two white, and one blue, into three distinct cells.

Let $D = \{r_1, r_2, r_3, w_1, w_2, b\}$

Let $R = \{c_1, c_2, c_3\}$.

$$G = \left\{ \begin{pmatrix} r_1r_2r_3w_1w_2b \\ r_1r_2r_3w_1w_2b \end{pmatrix}, \begin{pmatrix} r_1r_2r_3w_1w_2b \\ r_1r_3r_2w_1w_2b \end{pmatrix}, \begin{pmatrix} r_1r_2r_3w_1w_2b \\ r_2r_1r_3w_1w_2b \end{pmatrix}, \begin{pmatrix} r_1r_2r_3w_1w_2b \\ r_2r_3r_1w_1w_2b \end{pmatrix}, \begin{pmatrix} r_1r_2r_3w_1w_2b \\ r_3r_1r_2w_1w_2b \end{pmatrix}, \begin{pmatrix} r_1r_2r_3w_1w_2b \\ r_3r_2r_1w_1w_2b \end{pmatrix} \right.$$

$$\left. \begin{pmatrix} r_1r_2r_3w_1w_2b \\ r_1r_2r_3w_2w_1b \end{pmatrix}, \begin{pmatrix} r_1r_2r_3w_1w_2b \\ r_1r_3r_2w_2w_1b \end{pmatrix}, \begin{pmatrix} r_1r_2r_3w_1w_2b \\ r_2r_1r_3w_2w_1b \end{pmatrix}, \begin{pmatrix} r_1r_2r_3w_1w_2b \\ r_2r_3r_1w_2w_1b \end{pmatrix}, \begin{pmatrix} r_1r_2r_3w_1w_2b \\ r_3r_1r_2w_2w_1b \end{pmatrix}, \begin{pmatrix} r_1r_2r_3w_1w_2b \\ r_3r_2r_1w_2w_1b \end{pmatrix} \right\}$$

$$P_G = \frac{1}{12}\{x_1^6 + 4x_1^4x_2 + 2x_1^3x_3 + 3x_1^2x_2^2 + 2x_1x_2x_3\}$$

The number of equivalence classes of function from D to R is

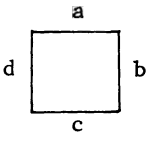
$$P_G(3,3,3) = \frac{1}{12}\{3^6 + 4\cdot3^4\cdot3 + 2\cdot3^3\cdot3 + 3\cdot3^2\cdot3^2 + 2\cdot3\cdot3\cdot3\} = 180$$

□

5-13. The sides of a square are colored using three colors. Two colorings are said to be equivalent if one can be obtained from the other by a rotation of the square and/or by a permutation of the colors. Find the number of equivalence classes of the coloring schemes.

$D = \{a,b,c,d\}$

$R = \{1,2,3\}$



$G = \{\binom{abcd}{abcd}, \binom{abcd}{bcda}, \binom{abcd}{cdab}, \binom{abcd}{dabc}\}$

$P_G = \frac{1}{4}(x_1^4 + x_2^2 + 2x_4)$

$H = \{\binom{123}{123}, \binom{123}{132}, \binom{123}{213}, \binom{123}{231}, \binom{123}{312}, \binom{123}{321}\}$

$P_H = \frac{1}{6}(x_1^3 + 3x_1x_2 + 2x_3)$

Applying Theorem 5-7, the number of equivalence classes of functions from D to R is

$$\frac{1}{4\cdot 6}\left(\frac{\partial^4}{\partial z_1^4} + \frac{\partial^2}{\partial z_2^2} + 2\frac{\partial}{\partial z_4}\right)\left(e^{3(z_1+z_2+z_3+z_4)} + 3e^{(z_1+z_2+z_3+z_4)} + 2(z_2+z_4) + 2e^{3z_3}\right)\Bigg|_{z_1 = z_2 = z_3 = z_4 = 0}$$

$$= \frac{1}{24}[3^4 + 3 + 3^2 + 3\cdot 3^2 + 2(3 + 3\cdot 3)]$$

$$= 6$$

□

5-14. (a) The cells of a 4×4 chessboard are to be painted with white and black paints. To generate all the black and white patterns, how many distinct drawing do we have to make?

(b) If each drawing can be used either as a pattern or as the negative of a pattern, how many distinct drawings do we have to make?

(a) Let D be the set of 16 cells.

Let $R = \{b,w\}$.

Let G be the permutation group on D containing permutations corresponding to each of the 4 possible rotations of the chessboard, specifically

(i) The identity permutation - x_1^{16}

(ii) 2 permutations corresponding to $\pm 90^\circ$ rotations - x_4^4

(iii) The permutation corresponding to a 180° rotation - x_2^8

$$P_G = \frac{1}{4}(x_1^{16} + 2x_4^4 + x_2^8)$$

$$P_G(2,2,2,2) = \frac{1}{4}(2^{16} + 2\cdot 2^4 + 2^8) = 16{,}456$$

(b) $H = \{\binom{bw}{bw}, \binom{bw}{wb}\}$

$$P_H = \frac{1}{2}(x_1^2 + x_2)$$

With both G acting on D and H acting on R, the number of equivalence classes of functions is

$$\frac{1}{4}\cdot\frac{1}{2}\left(\frac{\partial^{16}}{\partial z_1^{16}} + 2\frac{\partial^4}{\partial z_4^4} + \frac{\partial^8}{\partial z_2^8}\right)\left(e^{2(z_1+z_2+z_3+z_4)} + e^{2(z_2+z_4)}\right)\Bigg|_{z_1 = z_2 = z_3 = z_4 = 0}$$

$$= \frac{1}{8}[2^{16} + 2\cdot 2^4 + 2^8 + 2\cdot 2^4 + 2^8) = 8264$$

□

5-15. An electronic system consists of four different input units and four different output units. Connections between the various input units and output units are made

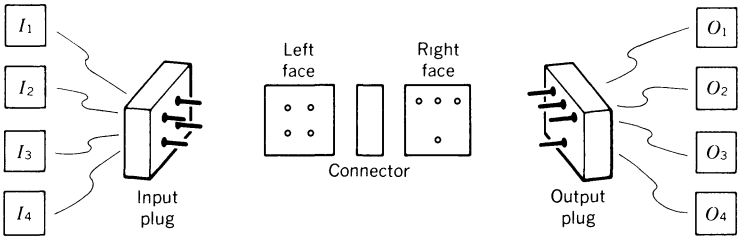


Figure 5P-2

as shown in Fig. 5P-2. Connections between its left and right terminals are fixed internally within the connector.

(a) If each input unit must be connected to a distinct output unit, how many different connectors will be needed to permit all such system connections?

(b) If each input unit must be connected to an output unit, but any number of input units may share the same output unit, how many different connectors will be needed to permit all such system connections?

(c) Suppose the output plug is redesigned as shown in Fig. 5P-3 with a corresponding modification to the right face of the connector. Repeat part (b), with these modifications.

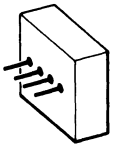


Figure 5P-3

Let $D = \{I_1, I_2, I_3, I_4\}$ and $R = \{O_1, O_2, O_3, O_4\}$

In each of the three parts of the problem, the number of connectors required is equal to the number of equivalence classes of functions from D to R, under the stated conditions.

(a) The effect of the input plug's pin arrangement is to introduce a permutation group G on D containing 4 permutations, one for each rotation of the plug. Find the number of equivalence classes of one-to-one functions from D to R.

Let G' be a permutation group on the set of all 4! one-to-one functions from D to R. G' contains 4 permutations, one for each in G. $\pi' \in G'$, corresponding to $\pi \in G$, permutes function f_1 into function f_2 according to the rule

$$\pi'(f_1) = f_2 \quad \text{iff} \quad f_1[\pi(d)] = f_2(d) \quad \text{for all } d \in D$$

Only the identity permutation in G' contains invariances. Applying Burnside's theorem, the number of equivalence classes induced by G' is

$$\frac{1}{|G'|} \sum_{\pi' \in G'} \Psi(\pi') = \frac{1}{4} \cdot 4! = 6$$

(b) Find the number of equivalence classes of all function from D to R, with G acting on D.

$$P_G = \frac{1}{4}(x_1^4 + x_2^2 + 2x_4)$$

Applying Corollary 5-1, the number of equivalence classes is

$$P_G(4,4,4,4) = \frac{1}{4}(4^4 + 4^2 + 2 \cdot 4) = 70$$

(c) The modification to the output plug introduces a permutation group H on R.

$$P_H = \frac{1}{2}(x_1^4 + x_2^2)$$

The number of equivalence classes of functions from D to R with G acting on D and H acting on R is found by applying Theorem 5-7.

$$\frac{1}{4}\cdot\frac{1}{2}\left(\frac{\partial^4}{\partial z_1^4}+\frac{\partial^2}{\partial z_2^2}+2\frac{\partial}{\partial z_4}\right)\left(e^{4(z_1+z_2+z_4)}+e^{4(z_2+z_4)}\right)\Bigg|_{\substack{z_1=0\\ z_2=0\\ z_4=0}}$$

$$=\frac{1}{8}[4^4+4^2+4^2+2(4+4)]=38$$

□

5-16. From the 2^{2n} 2n-digit binary sequences, subsets of k sequences are selected. Two subsets are said to be indistinguishable if they become identical subsets when the sequences in one subset are read from left to right and the sequences in the other are read from right to left. Find the number of distinct subsets of sequences.

Let D be the set of 2^{2n} 2n-digit binary sequences.

Let R = {0,1}

Interpret functions from D to R as characteristic functions of subsets. A function f specifies a subset S according to the rule

$$d \in S \quad \text{iff} \quad f(d) = 1, \qquad \text{for all } d \in D$$

Let G be a permutation group on D containing the following two permutations with cycle structures as given.

(i) The identity permutation - $x_1^{2^{2n}}$

(ii) The permutation which permutes each 2n-tuple in D into the "reversed" 2n-tuple - $x_1^{2^n}\, x_2^{\frac{1}{2}(2^{2n}-2^n)}$

The cycle structure of the second permutation follows from the fact that 2^n of the 2^{2n} 2n-tuples are symmetric and are therefore left invariant by the permutation.

$$P_G=\frac{1}{2}\left(x_1^{2^{2n}}+x_1^{2^n}\,x_2^{2^{n-1}(2^n-1)}\right)$$

Let w(0) = 1

w(1) = y

Then the coefficient of y^k in the pattern inventory is the number of distinct k-subsets.

$$P_G(1+y,\ 1+y^2)=\frac{1}{2}\left[(1+y)^{2^{2n}}+(1+y)^{2^n}\,(1+y^2)^{2^{n-1}(2^n-1)}\right]$$

The coefficient of y^k is

$$\frac{1}{2}\left[\binom{2^{2n}}{k} + \sum_{j=0}^{[\frac{k}{2}]} \binom{2^n}{k-2j}\binom{2^{n-1}(2^n-1)}{j}\right]$$

□

5-17. To complement a binary digit means to change a 0 into a 1 and to change a 1 into a 0. Let D be the set of the 2^n n-digit binary sequences. Let G be a set of 2^n permutations of the set D, $\{\pi_{000\cdots0}, \pi_{000\cdots1}, \cdots, \pi_{111\cdots1}\}$, where the permutations are identified by the subscripts which are n-digit binary sequences. The permutation $\pi_{a_1a_2\cdots a_n}$ permutes an an n-digit binary sequence into another one by complementing its digits that correspond to the 1's in $a_1a_2\cdots a_n$.

(a) Show that G is a permutation group.

(b) Show that the cycle index of G is

$$1/2^n[x_1^{2^n} + (2^n-1)x_2^{2^{n-1}}]$$

(c) Let R = {0,1}. Find the number of equivalence classes of functions from D to R. (This is the number of distinct Boolean functions of n variables when complementation of the variables is permitted.)

(d) Find the number of equivalence classes of functions from D to R in which exactly k of the 2^n elements in D assume the value 1, for k being odd and for k being even.

(a) Let $\underline{a}_i$ denote the binary n-tuple $a_{i_1}, a_{i_2}, \cdots, a_{i_n}$. Let $\oplus$ represent the binary operation modulo 2 addition, defined by

$\oplus$	0	1
0	0	1
1	1	0

(i) For any $\pi_{\underline{a}_i}, \pi_{\underline{a}_j} \in G$, $\pi_{\underline{a}_i}\pi_{\underline{a}_j} \in G$ since $\pi_{\underline{a}_i}\pi_{\underline{a}_j} = \pi_{\underline{a}_i \oplus \underline{a}_j} = \pi_{\underline{a}_k}$ for some $\underline{a}_k$

(ii) For any $\pi_{\underline{a}_i}, \pi_{\underline{a}_j}, \pi_{\underline{a}_k} \in G$

$$\pi_{\underline{a}_i}(\pi_{\underline{a}_j}\pi_{\underline{a}_k}) = \pi_{\underline{a}_i \oplus (\underline{a}_j \oplus \underline{a}_k)} = \pi_{(\underline{a}_i \oplus \underline{a}_j) \oplus \underline{a}_k} = (\pi_{\underline{a}_i}\pi_{\underline{a}_j})\pi_{\underline{a}_k}$$

(iii) The identity permutation is $\pi_{00\cdots0}$.

(iv) Each permutation is its own inverse. Since

$$\pi_{\underline{a}_i}\pi_{\underline{a}_i} = \pi_{\underline{a}_i \oplus \underline{a}_i} = \pi_{00\cdots0}$$

(b) The cycle structure of $\pi_{00\cdots0}$ is $x_1^{2^n}$.

Any one of the remaining (2^n-1) permutations $\pi_{\underline{a}_i}$ has one or more 1's in its subscript. Let $\underline{d} \in D$. Then $\pi_{\underline{a}_i}(\underline{d}) \in D$ is distinct from $\underline{d}$. Furthermore,

$$\pi_{\underline{a}_i}(\pi_{\underline{a}_i}(\underline{d})) = \pi_{00\cdots0}(\underline{d}) = \underline{d}$$

so that each $\underline{d} \in D$ is involved in a cycle of length 2 in $\pi_{\underline{a}_i}$. There are 2^{n-1} such cycles, and the cycle structure of each of these remaining (2^n-1) permutations is $x_2^{2^{n-1}}$

$$P_G = \frac{1}{2^n}\left[x_1^{2^n} + (2^n-1)\, x_2^{2^{n-1}}\right]$$

(c)
$$P_G(2,2) = \frac{1}{2^n}\left[2^{2^n} + (2^n-1)\cdot 2^{2^{n-1}}\right]$$

(d) Let $w(0) = 1$

$w(1) = y$

The coefficient of y^k in the pattern inventory

$$P_G(1+y,\ 1+y^2) = \frac{1}{2^n}\left[(1+y)^{2^n} + (2^n-1)(1+y^2)^{2^{n-1}}\right]$$

is the number of equivalence classes in which exactly k of the elements in D assume the value 1.

k-odd

Only the first term in the pattern inventory contributes, giving

$$\frac{1}{2^n}\binom{2^n}{k}$$

as the coefficient of y^k.

k-even

Both terms contribute, giving

$$\frac{1}{2^n}\left[\binom{2^n}{k} + (2^n-1)\binom{2^{n-1}}{\frac{k}{2}}\right]$$

as the coefficient of y^k.

□

5-18. In Problem 5-17, find the number of equivalence classes of functions from D to R if there is also a permutation group $H = \{\binom{01}{01}, \binom{01}{10}\}$ acting on R.

$$P_H = \frac{1}{2}(x_1^2 + x_2)$$

The number of equivalence classes is

$$\frac{1}{2^n} \cdot \frac{1}{2} \left(\frac{\partial^{2^n}}{\partial z_1^{2^n}} + (2^n - 1) \frac{\partial^{2^{n-1}}}{\partial z_2^{2^{n-1}}} \right) \left(e^{2(z_1+z_2)} + e^{2z_2} \right) \Bigg|_{z_1 = z_2 = 0}$$

$$= \frac{1}{2^{n+1}} \left[2^{2^n} + (2^n - 1)(2^{2^{n-1}} + 2^{2^{n-1}}) \right]$$

$$= \frac{1}{2^{n+1}} \left[2^{2^n} + (2^n - 1) \cdot 2^{(2^{n-1}+1)} \right]$$

□

5-19. Two Boolean functions are considered to be equivalent if one can be obtained from the other by permuting and/or complementing the independent variables. (The definition of a Boolean function is given in Problem 1-17.) The set of all possible permutations and/or complementations of the independent variables can be represented by a permutation group G acting on the domain of the Boolean function.

(a) Find the number of equivalence classes of Boolean functions of two variables.

(b) Repeat part (a) for Boolean functions of three variables, given that the cycle index for G is

$$P_G = \frac{1}{48}(x_1^8 + 13x_2^4 + 8x_1^2x_3^2 + 8x_2x_6 + 6x_1^4x_2^2 + 12x_4^2)$$

(c) Repeat part (a) for Boolean functions of four variables, given that the cycle index for G is

$$P_G = \frac{1}{384}(x_1^{16} + 51x_2^8 + 48x_1^2x_2x_4^3 + 48x_8^2 + 12x_1^8x_2^4 + 84x_4^4 + 12x_1^4x_2^6 + 32x_1^4x_3^4 + 96x_2^2x_6^2)$$

(d) Repeat parts (a), (b), and (c), if two Boolean functions are considered to be equivalent if one can be obtained from the other by permuting and/or complementing the independent variables, and/or by complementing the value of the function.

(a) A Boolean function $F(y_1, y_2)$ of two variables y_1 and y_2 is a function from a domain $D = \{00, 01, 10, 11\}$ to a range $R = \{0,1\}$. The $2! \cdot 2^n$ combinations of a permutation and a complementation on these two variables are listed below, together with the corresponding permutation $\pi \in G$ acting on D. (A complemented variable is indicated by an overbar.)

$F(y_1, y_2)$	$\begin{pmatrix} 00 & 01 & 10 & 11 \\ 00 & 01 & 10 & 11 \end{pmatrix}$	x_1^4
$F(\bar{y}_1, y_2)$	$\begin{pmatrix} 00 & 01 & 10 & 11 \\ 10 & 11 & 00 & 01 \end{pmatrix}$	x_2^2
$F(y_1, \bar{y}_2)$	$\begin{pmatrix} 00 & 01 & 10 & 11 \\ 01 & 00 & 11 & 10 \end{pmatrix}$	x_2^2
$F(\bar{y}_1, \bar{y}_2)$	$\begin{pmatrix} 00 & 01 & 10 & 11 \\ 11 & 10 & 01 & 00 \end{pmatrix}$	x_2^2
$F(y_2, y_1)$	$\begin{pmatrix} 00 & 01 & 10 & 11 \\ 00 & 10 & 01 & 11 \end{pmatrix}$	$x_1^2 x_2$
$F(\bar{y}_2, y_1)$	$\begin{pmatrix} 00 & 01 & 10 & 11 \\ 10 & 00 & 11 & 01 \end{pmatrix}$	x_4
$F(y_2, \bar{y}_1)$	$\begin{pmatrix} 00 & 01 & 10 & 11 \\ 01 & 11 & 00 & 10 \end{pmatrix}$	x_4
$F(\bar{y}_2, \bar{y}_1)$	$\begin{pmatrix} 00 & 01 & 10 & 11 \\ 11 & 01 & 10 & 00 \end{pmatrix}$	$x_1^2 x_2$

Thus

$$P_G = \frac{1}{8}(x_1^4 + 2x_1^2x_2 + 3x_2^2 + 2x_4)$$

With G acting on D, the number of equivalence classes of function from D to R is

$$P_G(2,2,2,2) = \frac{1}{8}(2^4 + 2\cdot 2^2\cdot 2 + 3\cdot 2^2 + 2\cdot 2) = 6$$

(b) $$P_G(2,2,\cdots) = \frac{1}{48}(2^8 + 13\cdot 2^4 + 8\cdot 2^2\cdot 2^2 + 8\cdot 2\cdot 2 + 6\cdot 2^4\cdot 2^2 + 12\cdot 2^2) = 22$$

(c) $$P_G(2,2,\cdots) = \frac{1}{384}(2^{16} + 51\cdot 2^8 + 48\cdot 2^2\cdot 2\cdot 2^3 + 48\cdot 2^2 + 12\cdot 2^8\cdot 2^4 + 84\cdot 2^4 + 12\cdot 2^4\cdot 2^6 + 32\cdot 2^4\cdot 2^4 + 96\cdot 2^2\cdot 2^2) = 402$$

(d) The permutation group $H = \{\begin{pmatrix} 01 \\ 01 \end{pmatrix}, \begin{pmatrix} 01 \\ 10 \end{pmatrix}\}$ acting on R is introduced.

$$P_H = \frac{1}{2}(x_1^2 + x_2)$$

Determine the number of equivalence classes or functions from D to R induced by G acting on D and H acting on R.

For functions of two variables, this is

$$\frac{1}{8}\left(\frac{\partial^4}{\partial z_1^4} + 2\frac{\partial^2}{\partial z_1^2}\frac{\partial}{\partial z_2} + 3\frac{\partial^2}{\partial z_2^2} + 2\frac{\partial}{\partial z_4}\right)\frac{1}{2}\left(e^{2(z_1+z_2+z_3+z_4)} + e^{2(z_2+z_4)}\right)\Bigg|_{z_1,\, z_2,\, z_3,\, z_4 = 0}$$

$$= \frac{1}{16}(2^4 + 2\cdot 2^2\cdot 2 + 3\cdot 2^2 + 2\cdot 2 + 0 + 0 + 3\cdot 2^2 + 2\cdot 2)$$

$$= 4$$

For functions of three variables, this is

$$\frac{1}{48}\left(\frac{\partial^8}{\partial z_1^8} + 13\frac{\partial^4}{\partial z_2^4} + 8\frac{\partial^2}{\partial z_1^2}\frac{\partial^2}{\partial z_3^2} + 8\frac{\partial}{\partial z_2}\frac{\partial}{\partial z_6} + 6\frac{\partial^4}{\partial z_1^4}\frac{\partial^2}{\partial z_2^2} + 12\frac{\partial^2}{\partial z_4^2}\right) \cdot \frac{1}{2}\left(e^{2(z_1+z_2+\cdots+z_6)} + e^{2(z_2+z_4+z_6)}\right) \Bigg|_{z_1 = z_2 = \cdots = z_6 = 0}$$

$$= \frac{1}{96}(2^8 + 13\cdot 2^4 + 8\cdot 2^2\cdot 2^2 + 8\cdot 2\cdot 2 + 6\cdot 2^4\cdot 2^2 + 12\cdot 2^2 + 0 + 13\cdot 2^4 + 0 + 8\cdot 2\cdot 2 + 0 + 12\cdot 2^2)$$

$$= 14$$

For functions of four variables, this is

$$\frac{1}{384}\left(\frac{\partial^{16}}{\partial z_1^{16}} + 51\frac{\partial^8}{\partial z_2^8} + 48\frac{\partial^2}{\partial z_1^2}\frac{\partial}{\partial z_2}\frac{\partial^3}{\partial z_4^3} + 48\frac{\partial^2}{\partial z_8^2} + 12\frac{\partial^8}{\partial z_1^8}\frac{\partial^4}{\partial z_2^4} + 84\frac{\partial^4}{\partial z_4^4} + 12\frac{\partial^4}{\partial z_1^4}\frac{\partial^6}{\partial z_2^6} + 32\frac{\partial^4}{\partial z_1^4}\frac{\partial^4}{\partial z_3^4} + 96\frac{\partial^2}{\partial z_2^2}\frac{\partial^2}{\partial z_6^2}\right) \cdot \frac{1}{2}\left(e^{2(z_1+z_2+\cdots+z_8)} + e^{2(z_2+z_4+z_6+z_8)}\right) \Bigg|_{z_1 = z_2 = \cdots = z_8 = 0}$$

$$= \frac{1}{384\cdot 2}(2^{16} + 51\cdot 2^8 + 48\cdot 2^2\cdot 2\cdot 2^3 + 48\cdot 2^2 + 12\cdot 2^8\cdot 2^4 + 84\cdot 2^4 + 12\cdot 2^4\cdot 2^6 + 32\cdot 2^4\cdot 2^4 + 96\cdot 2^2\cdot 2^2 + 0 + 51\cdot 2^8 + 0 + 48\cdot 2^2 + 0 + 84\cdot 2^4 + 0 + 0 + 96\cdot 2^2\cdot 2^2) = 222$$

□

5-20. Let S be a finite set of elements, and let G be a permutation group of S. Two subsets A and B of S are said to be equivalent if for some permutation π in G, the subset A is the set of all elements $\pi(b)$ obtained by letting b run through the elements in B.

(a) Show that all the subsets of S are partitioned into equivalence classes of subsets by the equivalence relation defined above.

(b) Show that the number of equivalence classes of subsets of S is equal to $P_G(2,2,\cdots,2)$. /

(a) For A, B $\subseteq$ S, $\pi \in G$, the notation

$$\pi : A \to B$$

denotes

$$B = \bigcup_{a \in A} \pi(a)$$

Reflexive — For all $A \subseteq S$ — where π_0 is the identity permutation in G

$$\pi_0 : A \to A$$

Symmetric — For all A, B $\subseteq$ S

if $\pi : A \to B$ $\quad \pi \in G$

then $\pi^{-1} : B \to A$ $\quad \pi^{-1} \in G$

Transitive — For all A, B, C $\subseteq$ S

if $\pi_1 : A \to B$ $\quad \pi_1 \in G$

and $\pi_2 : B \to C$ $\quad \pi_2 \in G$

then $\pi_2\pi_1 : A \to C$ $\quad \pi_2\pi_1 \in G$

(b) Let $R = \{0,1\}$

A 1 : 1 correspondence between functions f_A from S to R and subsets A of S is established by

$$s \in A \quad \text{iff} \quad f_A(s) = 1, \qquad \text{for all } s \in S$$

Further, for any A, B, $\subseteq$ S, with corresponding functions f_A, f_B

$$\pi : A \to B \quad \text{iff} \quad f_B(\pi(s)) = f_A(s) \qquad \text{for all } s \in S$$

The number of equivalence classes of subsets is equal to the number of equivalence classes of functions, which is given by

$$P_G(2,2,\cdots,2)$$

□

5-21. Let S be the set of all 2×2 matrices of 0's and 1's. Matrix multiplication is conducted in the field of integers modulo 2. Let T be a subset of S. A matrix t is in T if and only if t is non-singular (the inverse of t exists).

(a) Show that the set T, together with the operation of matrix multiplication in the field of integers modulo 2, forms a group.

(b) Define a binary relation as follows. An element s_1 in S is said to be related to s_2 in S if there exists a matrix t in T such that $s_1 = t^{-1} s_2 t$. Show that this relation is an equivalence relation.

(c) Find the number of equivalence relation defined above.

(a) A matrix t

$$t = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad a_{ij} \in \{0,1\}$$

is in T iff

$$|t| = a_{11}\, a_{22} + a_{21}\, a_{12} = 1$$

(i) The identity element is $t_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(ii) For any $t \in T$,

$$t = \begin{bmatrix} a_{11} & a_{22} \\ a_{21} & a_{22} \end{bmatrix},$$

$$t^{-1} = \begin{bmatrix} a_{22} & a_{12} \\ a_{21} & a_{11} \end{bmatrix}$$

since

$$t\, t^{-1} = \begin{bmatrix} a_{11}a_{22} + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{11} \\ a_{21}a_{22} + a_{22}a_{21} & a_{21}a_{12} + a_{22}a_{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = t_0$$

and $t^{-1} \in T$ since

$$|t^{-1}| = a_{22}a_{11} + a_{21}a_{12} = |t| = 1$$

(iii) For any $t_1, t_2 \in T$

$$|t_1| = 1, \quad |t_2| = 1$$

and since

$$|t_1 t_2| = |t_1|\, |t_2| = 1$$

$\therefore\ t_1 t_2 \in T$

(iv) Matrix multiplication is associative.

(b)

Reflexive $\quad s = t_0^{-1}\, s\, t_0 \qquad$ for all $s \in S$

Symmetric $\quad$ if $s_1 = t^{-1}\, s_2 t \qquad t \in T$

then $t\, s_1\, t^{-1} = tt^{-1}\, s_2\, tt^{-1} = s_2$

$s_2 = (t^{-1})^{-1}\, s_1 t^{-1} \qquad t^{-1} \in T$

Transitive

if $\quad s_1 = t_x^{-1}\, s_2\, t_x$

$s_2 = t_y^{-1}\, s_3\, t_y \qquad t_x, t_y \in T$

Then $\quad s_1 = t_x^{-1}\, t_y^{-1}\, s_3\, t_y\, t_x$

$= (t_y t_x)^{-1}\, s_3\, t_y\, t_x \qquad t_y t_x \in T$

(c) T contains 6 matrices

$$\underset{t_0}{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}} \quad \underset{t_1}{\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}} \quad \underset{t_2}{\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}} \quad \underset{t_3}{\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}} \quad \underset{t_4}{\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}} \quad \underset{t_5}{\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}}$$

Let $G = \{\pi_0, \pi_1, \cdots, \pi_5\}$ be a permutation group on S defined by

$$\pi_i(s) = t_i^{-1}\, s\, t_i \qquad \text{for all } s \in S, \qquad i = 0,1,\cdots, 5$$

$\Psi(\pi_i)$ is determined by computing the number of arbitrary choices for the a_{ij} in the matrix equation

$$\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} = \begin{bmatrix} & \\ t_i^{-1} & \\ & \end{bmatrix} \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} \begin{bmatrix} & \\ t_i & \\ & \end{bmatrix}$$

This gives

$$\Psi(\pi_0) = 16, \quad \Psi(\pi_1) = \Psi(\pi_2) = \cdots = \Psi(\pi_5) = 4$$

Applying Burnside's theorem, the number of equivalence classes induced by G in S is

$$\frac{1}{|G|} \sum_{\pi \in G} \Psi(\pi) = \frac{1}{6}(16 + 4 + 4 + 4 + 4 + 4) = 6$$

□

5-22. Let G be a permutation group of a set D. Two permutations π_1 and π_2 in G are said to be related by the self-conjugate relation if there exists a permutation π in G which is such that $\pi_1 = \pi^{-1}\pi_2\pi$.

(a) Show that the self-conjugate relation is an equivalence relation. (The equivalence classes of permutations are called self-conjugate classes.)

(b) Prove that if π_1 and π_2 are permutations in the same self-conjugate class, the number of invariances under π_1 is equal to that under π_2; that is, $\Psi(\pi_1) = \Psi(\pi_2)$.

(c) Show that the number of equivalence classes into which D is divided by the equivalence relation induced by G is equal to

$$\frac{1}{|G|} \sum \Psi(\pi_K) \cdot |K|$$

where the summation is over all self-conjugate classes of permutations in G, π_K is any permutation in the self-conjugate class K, and $|K|$ is the number of permutations in K.

(a)

Reflexive — Let π_0 be the identity permutation,

then $\pi_1 = \pi_0^{-1}\,\pi_1\,\pi_0$ for all $\pi_1 \in G$

Symmetric — if $\pi_1 = \pi^{-1}\,\pi_2\,\pi$ $\pi \in G$

then $\pi\,\pi_1\pi^{-1} = \pi\,\pi^{-1}\,\pi_2\,\pi\,\pi^{-1} = \pi_2$

$\pi_2 = (\pi^{-1})^{-1}\,\pi_1\,\pi^{-1}$ $\pi^{-1} \in G$

Transitive — if $\pi_1 = \pi_x^{-1}\,\pi_2\,\pi_x$ and $\pi_2 = \pi_y^{-1}\,\pi_3\,\pi_y$

$\pi_x, \pi_y \in G$

then $\pi_1 = \pi_x^{-1}\,\pi_y^{-1}\,\pi_3\,\pi_y\,\pi_x$

$= (\pi_y\pi_x)^{-1}\,\pi_3\,\pi_y\,\pi_x$ $\pi_y\pi_x \in G$

(b)

$\pi_1 = \pi^{-1}\,\pi_2\,\pi$

if $\pi_1(a) = a$

then $\pi^{-1}\,\pi_2\,\pi(a) = a$

$\pi\,\pi^{-1}\,\pi_2\,\pi(a) = \pi(a)$

$\pi_2(\pi(a)) = \pi(a)$

$\pi_2 = (\pi^{-1})^{-1}\,\pi_1\,\pi^{-1}$

if $\pi_2(a) = a$

then $(\pi^{-1})^{-1}\,\pi_1\,\pi^{-1}(a) = a$

$\pi^{-1}\,(\pi^{-1})^{-1}\,\pi_1\,\pi^{-1}(a) = \pi^{-1}(a)$

$\pi_1(\pi^{-1}(a)) = \pi^{-1}(a)$

Thus there is a 1:1 correspondence between invariances of π_1 and π_2 and

$$\Psi(\pi_1) = \Psi(\pi_2)$$

(c) Using Burnside's theorem, the number of equivalence classes induced by G is

$$\frac{1}{|G|} \sum_{\pi \in G} \Psi(\pi) = \frac{1}{|G|} \sum_{K} \sum_{\pi \in K} \Psi(\pi) = \frac{1}{|G|} \sum_{K} \Psi(\pi_K) \cdot |K|$$

□

5-23. A group G is said to be a cyclic group if every element in the group is a power of some fixed element π in G. An element π_i is said to be of power j if j is the smallest positive integer such that $\pi_i = \pi^j$. If G contains n elements, it can be shown that these n elements can be expressed as $\pi, \pi^2, \pi^3, \cdots, \pi^{n-1}, \pi^n$, where π^n is the identity element in G.

(a) Let G be a cyclic permutation group of a set S. Show that if there are n permutations in G, the number of equivalence classes into which the set S is divided by the equivalence relation induced by G is

$$\frac{1}{n} \sum_{d|n} \Psi(\pi_d)\phi\left(\frac{n}{d}\right)$$

where the summation is over all divisors d of n, π_d is the element in G of power d, and $\phi(n/d)$ is the number of positive integers which are less than or equal to n/d and are prime to n/d. [Recall that $\phi(x)$ is called the Euler's function. See Problem 4-11.]

(b) Find the number of distinct necklaces made up of eight beads of three distinct colors when cyclic rotations of the necklaces are allowed.

(a)

Let $G = \{\pi_1, \pi_2, \cdots, \pi_n\}$

where the element π_k is equal to π_1^k, $k = 1,2,\cdots,n$, and π_n is the identity element.

For any divisor d of n, the permutations

$$\{\pi_d, \pi_d^2, \cdots, \pi_d^{\frac{n}{d}}\}$$

form a cyclic subgroup of order $\frac{n}{d}$.

Consider π_d^k, where $k \in \{1,2,\cdots, \frac{n}{d}\}$

and $(k, \frac{n}{d}) = 1$ ((a,b) will denote the greatest common divisor of a and b)

$\Psi(\pi_d^{\,k}) \geq \Psi(\pi_d)$ since every invariance of π_d is an invariance of $\pi_d^{\,k}$

Assume $\Psi(\pi_d^{\,k}) > \Psi(\pi_d)$.

Then π_d must have a cycle of length $\ell > 1$ such that $\ell \mid k$, and since $(k, \frac{n}{d}) = 1$,

$$\ell \nmid \frac{n}{d}$$

But all cycle lengths of π_d must divide $\frac{n}{d}$, since

$$\pi_d^{\frac{n}{d}} = \pi_1^{\,n}, \qquad \text{the identity permutation.}$$

Therefore,

$$\Psi(\pi_d^{\,k}) = \Psi(\pi_d) \qquad \text{for all } k \in \{1,2,\cdots, \tfrac{n}{d}\} \text{ such that } k \text{ is prime to } \tfrac{n}{d}.$$

and $\Psi(\pi_d) \cdot \phi(\frac{n}{d})$ counts the invariances in $\phi(\frac{n}{d})$ of the permutations in G.

By summing over all divisors d of n, invariances in all permutations of G are counted. This may be shown in two steps.

(i) No permutation has its invariances counted twice.

If $d_1 \mid n$ and $d_2 \mid n$, $d_1 \neq d_2$,

and $k_1 \in \{1,2,\cdots, \frac{n}{d_1}\}$, $k_2 \in \{1,2,\cdots, \frac{n}{d_2}\}$,

$$(k_1, \tfrac{n}{d_1}) = 1, \qquad (k_2, \tfrac{n}{d_2}) = 1,$$

then

$$\pi_{d_1}^{\,k_1} \neq \pi_{d_2}^{\,k_2}.$$

Assume $\pi_{d_1}^{\,k_1} = \pi_{d_2}^{\,k_2}$.

Then $(\pi_1^{\,d_1})^{k_1} = (\pi_1^{\,d_2})^{k_2}$

$$k_1 d_1 = k_2 d_2$$

$$\frac{n/d_1}{k_1} = \frac{n/d_2}{k_2}$$

But since

$$\frac{n}{d_1} \neq \frac{n}{d_2}, \quad (k_1, \tfrac{n}{d_1}) = 1, \quad (k_2, \tfrac{n}{d_2}) = 1$$

this is a contradiction.

(ii) Every permutation has its invariances counted.

For any $\pi_k \in G$, $k = 1,2,\cdots,n$

$\Psi(\pi_k)$ is counted in the term

$$\Psi(\pi_{(k,n)})\cdot\phi\left(\frac{n}{(k,n)}\right)$$

since

$$(k,n) \mid n$$

$$k/(k,n)$$

$$\pi_k = \pi_{(k,n)}$$

and $\left(\frac{k}{(k,n)}, \frac{n}{(k,n)}\right) = 1$

Finally, applying Burnside's theorem, the number of equivalence classes induced by G is

$$\frac{1}{|G|}\sum_{\pi \in G}\Psi(\pi) = \frac{1}{n}\sum_{d \mid n}\Psi(\pi_d)\cdot\phi\left(\frac{n}{d}\right)$$

(b) Let S be the set of 3^8 necklaces, where rotational equivalence is ignored.

Let $G = \{\pi_1, \pi_2, \cdots, \pi_8\}$ be a cyclic permutation group of order 8 on S. π_k, $k = 1,2,\cdots,8$, permutes elements of S into those obtained from the former by a clockwise rotation of k bead positions.

The number of equivalence classes induced by G on S is

$$\frac{1}{8}\sum_{d \mid 8}\Psi(\pi_d)\ \phi\ \left(\frac{8}{d}\right) =$$

$$\frac{1}{8}[\Psi(\pi_1)\ \phi(8) + \Psi(\pi_2)\ \phi(4) + \Psi(\pi_4)\ \phi(2) + \Psi(\pi_8)\ \phi(1)]$$

$$= \frac{1}{8}[3\cdot 4 + 3^2\cdot 2 + 3^4\cdot 1 + 3^8\cdot 1] = 834$$

□

5-24. Consider a set D together with a permutation group G of D and a set R together with a permutation group H of R.

(a) Let π be a permutation in G, and let τ be a permutation in H. Let f be a function in the set of functions from D to R. Show that if

$$f(\pi(d)) = \tau f(d) \qquad \text{for all } d \in D$$

then

$$f(\pi^j(d)) = \tau^j f(d) \qquad \text{for all } d \in D;\ j = 1, 2, \cdots$$

(b) Let f be a one-to-one function. If $f(\pi(d)) = \tau f(d)$ for all d in D, prove that f maps any element in D that belongs to a cycle of length j in π into an element in R that belongs to a cycle of length j in τ.

(c) Let π in G be a permutation of the type $\{b_1, b_2, \cdots, b_{|D|}\}$ where b_j is the number of cycles of length j in π, and let τ in H be a permutation of the type $\{c_1, c_2, \cdots, c_{|R|}\}$ where c_j is the number of cycles of length j in τ. Show that the number of one-to-one functions f which are such that $f(\pi(d)) = \tau f(d)$ for all d in D is equal to

$$\prod_{j=1}^{|D|} j^{b_j} P(c_j, b_j) \qquad \text{where } P(c_j, b_j) = \begin{cases} \dfrac{c_j!}{(c_j - b_j)!} & c_j \geq b_j \\ 0 & c_j < b_j \end{cases}$$

Then show that

$$\prod_{j=1}^{|D|} j^{b_j} P(c_j, b_j) = \left[\left(\frac{\partial}{\partial z_1}\right)^{b_1} \left(\frac{\partial}{\partial z_2}\right)^{b_2} \cdots\right] \left[(1+z_1)^{c_1} (1+2z_2)^{c_2} \cdots\right]$$

evaluated at $z_1 = z_2 = \cdots = 0$.

(d) Show that the number of equivalence classes of one-to-one functions from D to R, with equivalence induced by the permutation groups G and H, is equal to

$$P_G\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \cdots\right) P_H(1+z_1, 1+2z_2, 1+3z_3, \cdots)$$

evaluated at $z_1 = z_2 = z_3 = \cdots = 0$.

(e) Show that if $|R| = |D|$, the number of equivalence classes of one-to-one functions from D to R is equal to

$$P_G\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \cdots\right) P_H(z_1, 2z_2, 3z_3, \cdots)$$

which in turn is equal to

$$P_H\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \cdots\right) P_G(z_1, 2z_2, 3z_3, \cdots)$$

(a) If $f(\pi(d)) = \tau f(d)$ for all $d \in D$

then $f(\pi^2(d)) = \tau f(\pi(d)) = \tau^2 f(d)$

$f(\pi^3(d)) = \tau^2 f(\pi(d)) = \tau^3 f(d)$ for all $d \in D$

$\vdots \qquad\qquad \vdots$

$f(\pi^j(d)) = \tau^{j-1} f(\pi(d)) = \tau^j f(d)$

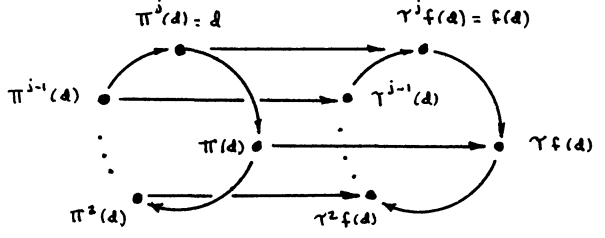
(b) The cycle $(d, \pi(d), \pi^2(d), \cdots, \pi^{j-1}(d))$ is mapped by f into $(f(d), \tau f(d), \tau^2 f(d), \cdots, \tau^{j-1} f(d))$. Since f is one-to-one, each of these range elements is distinct. Furthermore, since

$$\pi^j(d) = d$$

$$\tau^j f(d) = f(\pi^j(d)) = f(d)$$

Thus f(d) belongs to a cycle of length j in P.

(c) In constructing such a function f, the j domain elements in each cycle of length j in π, $j = 1,2,\cdots, |D|$, must be mapped onto the j range elements of a distinct cycle of length j in τ, in one of the j ways indicated below.



There are $P(c_j, b_j)$ ways to establish a one-to-one correspondence between all the cycles of length j in π and a subset of those in τ. For each such correspondence there are j ways to assign function values to domain elements in each cycle of length j, and thus j^{b_j} ways to assign function values to domain elements in all cycles of length j. Allowing j to vary from 1 to $|D|$ gives

$$\prod_{j=1}^{|D|} j^{b_j} \cdot P(c_j, b_j)$$

as the number of functions f satisfying the stated conditions. Since

$$j^{b_j} P(c_j, b_j) = \frac{d^{b_j}}{dz^{b_j}} (1 + jz)^{c_j} \Bigg|_{z = 0}$$

this can be rewritten as

$$\left(\frac{\partial^{b_1}}{\partial z_1^{b_1}} \frac{\partial^{b_2}}{\partial z_2^{b_2}} \cdots \frac{\partial^{b_{|D|}}}{\partial z_{|D|}^{b_{|D|}}}\right)\left((1+z_1)^{c_1} (1+2z_2)^{c_2} \cdots (1 + |R| z_{|R|})^{c_{|R|}}\right) \Bigg|_{z_1 = z_2 = \cdots = 0}$$

(d)

Let $G \times H$ be the permutation group on the set S of $P(|R|, |D|)$ one-to-one functions from D to R, containing $|G| \cdot |H|$ permutations denoted by (π, τ), $\pi \in G$, $\tau \in H$, and defined by

$$(\pi, \tau)\, f_1 = f_2 \qquad \text{iff} \qquad f_2 = \tau^{-1} f_1(\pi(d)) \qquad \text{for all } d \in D$$

Applying Burnside's theorem, the number of equivalence classes in S induced by $G \times H$ is

$$\frac{1}{|G \times H|} \sum_{(\pi, \tau) \in G \times H} \Psi(\pi, \tau)$$

$\Psi(\pi, \tau)$ was computed in part (c).

$$\frac{1}{|G \times H|} \sum_{(\pi, \tau) \in G \times H} \Psi(\pi, \tau) =$$

$$\frac{1}{|G| \cdot |H|} \sum_{(\pi, \tau) \in G \times H} \left[\prod_{i=1}^{|D|} \frac{\partial^{b_i}}{\partial z_i^{b_i}} \right] \left[\prod_{j=1}^{|R|} (1+jz_j)^{c_j} \right] \Bigg|_{z_1 = z_2 = \cdots = 0}$$

(the first bracketed factor depends only on $\pi \in G$; the second depends only on $\tau \in H$)

$$= \frac{1}{|G|} \sum_{\pi \in G} \left[\prod_{i=1}^{|D|} \frac{\partial^{b_i}}{\partial z_i^{b_i}} \right] \cdot \frac{1}{|H|} \sum_{\tau \in H} \left[\prod_{j=1}^{|R|} (1+jz_j)^{c_j} \right] \Bigg|_{z_1 = z_2 = \cdots = 0}$$

$$= P_G\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \cdots \frac{\partial}{\partial z_{|D|}}\right) \; P_H(1+z_1, 1+2z_2, \cdots, 1+|R| \cdot z_{|R|}) \Big|_{z_1 = z_2 = \cdots = 0}$$

(e) Let $k = |D| = |R|$

For any $(\pi, \tau) \in G \times H$,

$$\Psi(\pi, \tau) > 0 \qquad \text{iff} \qquad b_i \le c_i \qquad i = 1,2,\cdots,k$$

But, since

$$b_1 + 2b_2 + \cdots + kb_k = k = c_1 + 2c_2 + \cdots + kc_k$$

and

$$b_i,\ c_i \ge 0 \qquad i = 1,2,\cdots,k$$

it follows that

$$\Psi(\pi, \tau) > 0 \qquad \text{iff} \qquad b_i = c_i \qquad i = 1,2,\cdots,k$$

From part (c),

$$\Psi(\pi, \tau) = \begin{cases} \prod_{j=1}^{k} j^{b_j} \cdot P(b_j, b_j) = \prod_{j=1}^{k} j^{b_j} \cdot b_j! & \text{if } b_j = c_j, \quad j = 1,2,\cdots,k \\ 0 & \text{otherwise} \end{cases}$$

Since

$$j^{b_j} \cdot b_j! = \left(\frac{d}{dz}\right)^{b_j} (jz)^{b_j}$$

$\Psi(\pi, \tau)$ may be rewritten as

$$\Psi(\pi, \tau) = \left(\left(\frac{\partial}{\partial z_1}\right)^{b_1} \left(\frac{\partial}{\partial z_2}\right)^{b_2} \cdots \left(\frac{\partial}{\partial z_k}\right)^{b_k}\right)\left(z_1^{c_1} (2z_2)^{c_2} \cdots (kz_k)^{c_k}\right) \qquad [1]$$

which is non-zero iff $b_j = c_j$, $j = 1,2,\cdots,k$, as shown above.

Alternatively,

$$\Psi(\pi, \tau) = \left(\left(\frac{\partial}{\partial z_1}\right)^{c_1} \left(\frac{\partial}{\partial z_2}\right)^{c_2} \cdots \left(\frac{\partial}{\partial z_k}\right)^{c_k}\right)\left(z_1^{b_1} (2z_2)^{b_2} \cdots (kz_k)^{b_k}\right) \qquad [2]$$

Using [1] and [2] in

$$\frac{1}{|G \times H|} \sum_{(\pi, \tau) \in G \times H} \Psi(\pi, \tau)$$

gives

$$P_G\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \cdots, \frac{\partial}{\partial z_k}\right) \; P_H(z_1, 2z_2, \cdots, kz_k)$$

and

$$P_H\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \cdots, \frac{\partial}{\partial z_k}\right) \; P_G(z_1, 2z_2, \cdots, kz_k)$$

respectively.

□

5-25. Let D and R be two finite sets. Let f be a one-to-one function from D to R. The set of ordered pairs (d,f(d)) for all d in D is called a labeled set of R. Clearly, the labeled set is just an alternative representation of the one-to-one function f. Let H be a permutation group of R.

(a) Show that the number of equivalence classes of labeled sets of R is equal to

$$\left(\frac{d}{dz}\right)^{|D|} P_H(1 + z, 1, 1, \cdots) \Big|_{z = 0}$$

where equivalence between labeled sets is defined as equivalence between the corresponding one-to-one functions from D to R when the permutation group acting on D consists of only the identity permutation and the permutation group acting on R is H.

(b) If H also consists of the identity element only, find the number of equivalence classes of labeled sets of R.

(c) Let R be the set of the faces of a cube, and let H be the group of permutations of the faces obtained by the rotation of the cube. Let D be the set of integers, 1, 2, 3, 4, 5, and 6. Find the number of equivalence classes of labeled sets of R. If the cube is a die, what is the physical significance of the labeled sets?

(a) Let $\hat{H}$ be a permutation group on the set S of all one-to-one functions from D to R. For each $\tau \in H$ there is a $\hat{\tau} \in \hat{H}$ defined by

$$\hat{\tau}(f_1) = f_2 \qquad f_1, f_2 \in S$$

where

$$f_2(d) = \tau(f_1(d)) \qquad \text{for all } d \in D$$

$\hat{H}$ induces the required equivalence relation on S. The number of equivalence classes may be found using Burnside's theorem. It remains only to evaluate $\Psi(\hat{\tau})$, for each $\hat{\tau} \in \hat{H}$.

If $\hat{\tau}(f) = f$

then $\tau(f(d)) = f(d)$ for all $d \in D$

and conversely.

In words, for a one-to-one function, f to be invariant under $\hat{\tau} \in \hat{H}$, f must map elements in D only into elements of R which are invariant under $\tau \in H$. There are $\Psi(\tau)$ such range elements.

Evaluate $\Psi(\hat{\tau})$ by counting the ways in which one-to-one functions from D to R may be constructed using only the $\Psi(\tau)$ range elements invariant under τ.

$$\Psi(\hat{\tau}) = P(\Psi(\tau), |D|)$$
$$= \Psi(\tau)(\Psi(\tau)-1)(\Psi(\tau)\cdot 2) \cdots (\Psi(\tau) - |D| + 1)$$

Using Burnside's theorem, the number of equivalence classes in S induced by $\hat{H}$ is

$$\frac{1}{|\hat{H}|} \sum_{\hat{\tau} \in \hat{H}} \Psi(\hat{\tau}) = \frac{1}{|H|} \sum_{\tau \in H} P(\Psi(\tau), |D|)$$

which can be rewritten as

$$\frac{1}{|H|}\sum_{\tau\in H}\frac{d^{|D|}}{dz^{|D|}}(1+z)^{\Psi(\tau)}\Bigg|_{z=0} = \frac{d^{|D|}}{dz^{|D|}}\frac{1}{|H|}\sum_{\tau\in H}(1+z)^{\Psi(\tau)}\Bigg|_{z=0}$$

$$= \frac{d^{|D|}}{dz^{|D|}}P_H(1+z, 1, 1, \cdots)\Bigg|_{z=0}$$

(b) $P_H = x_1^{|R|}$

The number of equivalence classes is

$$\frac{d^{|D|}}{dz^{|D|}}(1+z)^{|R|}\Bigg|_{z=0} = |R|\ (|R|-1)(|R|-2)\cdots(|R|-|D|+1) = P(|R|, |D|)$$

(c) $P_H = \frac{1}{24}(x_1^6 + 3x_1^2x_2^2 + 6x_1^2x_4 + 6x_2^3 + 8x_3^3)$

The number of equivalence classes of labeled sets of R is

$$\frac{1}{24}\frac{d^6}{dz^6}[(1+z)^6 + 3(1+z)^2 + 6(1+z)^2 + 6 + 8]\Bigg|_{z=0}$$

$$= \frac{6!}{24} = 30$$

A labeled set is a way of assigning values to the faces of a die. Thus there are 30 possible dice.

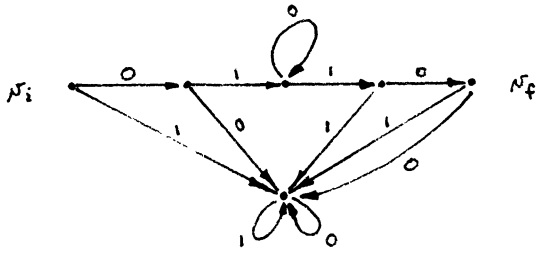
□

6-1. Show that in a finite graph with exactly two vertices of odd degree there is a path connecting these two vertices.

If G is connected there is an Euler path connecting these two vertices, by Theorem 6-3. Otherwise, these two vertices lie in a single connected component of G, by Theorem 6-2, and are connected by an Euler path for that component.

□

6-2. An electronic circuit is built to recognize sequences of 0's and 1's. In particular it shall accept sequences of the form 010^*10, where 0^* means any number (including none) of 0's. For example, 0110, 01010, and 01000010 are all acceptable sequences. Construct a directed graph in which every vertex has two outgoing edges labeled 0 and 1, and in which there are two vertices v_i (the initial vertex) and v_f (the final vertex) such that every path from v_i to v_f is a sequence of the form 010^*10. (For each sequence the circuit will start at v_i and trace the edges according to the 0's and 1's in the sequence. The circuit will accept the sequence if the path terminates at v_f.)



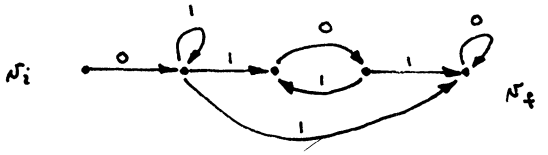
□

6-3. Repeat Problem 2 for sequences of the form $01^*(10)^*10^*$, where $(10)^*$ means any number (including none) of 10 patterns, using each of the following modifications:

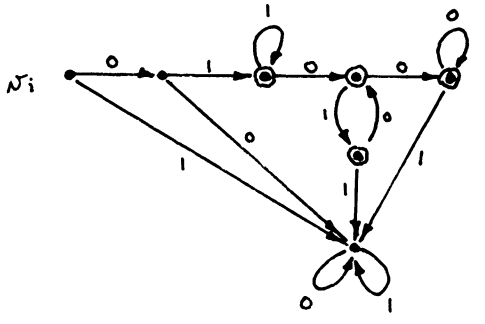
(a) For each vertex there is no restriction on the number of outgoing edges labeled 0 or 1. The sequence of labels on every path from v_i to v_f is a sequence of the form described above. Furthermore, for every sequence of the form described above there is a corresponding path from v_i to v_f.

(b) There may be several final vertices v_{f_1}, v_{f_2}, $\cdots$. The sequences of labels on every path from v_i to some final vertex v_{f_j} is a sequence of the form described above. Furthermore, for every sequence of the form described above there is a corresponding path from v_i to some final vertex v_{f_j}.

(a)



(b)



◎ indicates final vertex

□

6-4. Let V = {1,2,3,4}. Let D = {{1,2}, {1,3}, {1,4}, {2,3}, {2,4}, {3,4}}, that is, the set of all 2-subsets of the set V. Let G be the group consisting of all the permutations of the set V.

(a) Show that for each permutation π in G there is a corresponding permutation $\hat{\pi}$ on the set D such that $\hat{\pi}(\{a,b\}) = \{\pi(a), \pi(b)\}$.

(b) Show that the set of $\hat{\pi}$'s is a permutation group.

(c) Let R = {x,y} with w(x) = 1 and w(y) = y. Find the pattern inventory of the equivalence classes of functions from D to R.

(d) In terms of undirected graphs with four vertices, what is the significance of the result in part (c)?

(a) It is necessary only to show that

if $\{a,b\} \neq \{c,d\}$ $\quad a,b,c,d \in V$

Then $\hat{\pi}(\{a,b\}) \neq \hat{\pi}(\{c,d\})$.

Since π is a permutation on V,

if $v_1 \neq v_2$ $\quad v_1, v_2 \in V$

then $\pi(v_1) \neq \pi(v_2)$

It follows that

$$\hat{\pi}(\{a,b\}) = \{\pi(a),\ \pi(b)\} \neq \{\pi(c),\ \pi(d)\} = \hat{\pi}(\{c,d\})$$

(b)

(i) The identity permutation in $\hat{G}$ corresponds to the identity permutation in G.

(ii) The inverse of $\hat{\pi} \in \hat{G}$, corresponding to $\pi \in G$, is $\hat{\pi}^{-1} \in G$, corresponding to $\pi^{-1} \in G$.

$$\hat{\pi}^{-1}\,\hat{\pi}(\{a,b\}) = \hat{\pi}^{-1}\,(\{\pi(a),\ \pi(b)\}) = \{\pi^{-1}\pi(a),\ \pi^{-1}(b)\} = \{a,b\}$$

(iii) For any $\hat{\pi}_i,\ \hat{\pi}_j \in \hat{G}$, $\hat{\pi}_i\hat{\pi}_j \in \hat{G}$

$$\hat{\pi}_i\hat{\pi}_j\,(\{a,b\}) = \{\pi_i\pi_j(a),\ \pi_i\pi_j(b)\} = \{\pi_k(a),\ \pi_k(b)\} = \hat{\pi}_k\,(\{a,b\}) \qquad \hat{\pi}_k \in \hat{G}$$

(iv) Composition of permutations is associative.

(c)

$$P_G = \frac{1}{24}\{x_1^4 + 6x_1^2x_2 + 8x_1x_3 + 3x_2^2 + 6x_4\}$$

The cycle structure of each permutation in $\hat{G}$ can be inferred from the cycle structure of the corresponding permutation in G as follows:

π	$\hat{\pi}$
x_1^4	x_1^6
$x_1^2x_2$	$x_1^2x_2^2$
x_1x_3	x_3^2
x_2^2	$x_1^2x_2^2$
x_4	x_2x_4

$$P_{\hat{G}} = \frac{1}{24}\,(x_1^6 + 9x_1^2x_2^2 + 6x_2x_4 + 8x_3^2)$$

The pattern inventory is

$$\frac{1}{24}\,[(1+y)^6 + 9(1+y)^2\,(1+y^2)^2 + 6(1+y^2)(1+y^4) + 8(1+y^3)^2]$$

$$= 1 + y + 2y^2 + 3y^3 + 2y^4 + y^5 + y^6$$

(d) Each function f from D to R specifies an undirected graph on 4 vertices, without loops. $\hat{G}$ on D induces an equivalence relation on the set of all such graphs. Two graphs are equivalent iff they are isomorphic.

In the pattern inventory, the coefficient of y^k is the number of non-isomorphic, undirected graphs on 4 vertices having exactly k edges (and no loops).

□

6-5. Repeat Problem 4 for D being the set of all ordered pairs in V × V that contain two distinct elements.

Parts (a) and (b) are analogous to parts (a) and (b) of Problem 6-4, with ordered pairs replacing 2-subsets.

(c) The cycle structure of each permutation in $\hat{G}$ can be inferred from the cycle structure of the corresponding permutation in G as follows:

π	$\hat{\pi}$
x_1^4	x_1^{12}
$x_1^2 x_2$	$x_1^2 x_2^5$
$x_1 x_3$	x_3^4
x_2^2	x_2^6
x_4	x_4^3

$$P_{\hat{G}} = \frac{1}{24}(x_1^{12} + 6x_1^2 x_2^5 + 8x_3^4 + 3x_2^6 + 6x_4^3)$$

The pattern inventory is

$$\frac{1}{24}\left[(1+y)^{12} + 6(1+y)^2\,(1+y^2)^5 + 8(1+y^3)^4 + 3(1+y^2)^6 + 6(1+y^4)^3\right]$$

$$= 1 + y + 5y^2 + 13y^3 + 27y^4 + 38y^5 + 48y^6 + 38y^7 + 27y^8 + 13y^9 + 5y^{10} + y^{11} + y^{12}$$

(d) The coefficient of y^k in the pattern inventory is the number of non-isomorphic, directed graphs on 4 vertices having exactly k edges (and no loops).

□

6-6. Show that a linear graph G with no loops and n vertices is not bipartite if it has more than $[n^2/4]$ edges. (See Sec. 11-1 for a definition of bipartite graph).

Assume G is bipartite. Then there is a partition of the vertices of G into two subsets X and Y containing k and (n-k) vertices respectively, such that all edges of G connect vertices in X to vertices in Y. Since G is linear, the maximum number of edges is k(n-k). But,

$$k(n-k) \leq \begin{cases} \frac{n^2}{4} & n \text{ - even} \\ \frac{n^2-1}{4} & n \text{ - odd} \end{cases} = [\frac{n^2}{4}]$$

which is a contradiction.

□

6-7. Show that a linear graph with no loops and n vertices is connected if it has more than 1/2(n-1)(n-2) edges.

Assume G contains two connected components containing k and n-k vertices, $k = 1,2,\cdots, n-1$.

Since G is linear, the maximum number of edges in G is

$$\binom{k}{2} + \binom{n-k}{2}$$

But

$$\binom{k}{2} + \binom{n-k}{2} \leq \binom{n-1}{2} \qquad \text{for } k = 1,2,\cdots, n-1$$

which may be seen as follows.

Using the identity

$$\binom{a+b}{2} = \binom{a}{2} + \binom{b}{2} + ab$$

to obtain

$$\binom{n-1}{2} = \binom{k-1}{2} + \binom{n-k}{2} + (k-1)(n-k)$$

and

$$\binom{k-1}{2} = \binom{k}{2} - (k-1)$$

gives

$$\binom{n-1}{2} = \binom{k}{2} + \binom{n-k}{2} + (k-1)(n-k-1) \geq \binom{k}{2} + \binom{n-k}{2} \qquad \text{for } k = 1,2,\cdots, n-1$$

G contains more than $\binom{n-1}{2}$ edges, so this is a contradiction.

Similarly, G cannot contain more than two connected components by repeated application of the above argument.

Therefore, G must be connected.

□

6-8. An (undirected) complete graph is a graph in which there is an edge joining every two vertices.

(a) The edges of a complete graph with six vertices are to be painted either red or blue. Show that for any arbitrary way of painting the edges there is either a red triangle (all three edges are painted red) or a blue triangle in the graph.

(b) Show that among a group of six people there are either three who are mutual friends or three who are strangers to each other.

(a) Consider the 5 edges incident with any vertex v_x. Of these, 3 must be the same color, say red. If these 3 edges are $\{v_x, v_1\}$, $\{v_x, v_2\}$ and $\{v_x, v_3\}$, then either one of the edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_1\}$ is red, completing a red triangle, or they are all blue, forming a blue triangle.

(b) Let the vertices of the complete graph with 6 vertices correspond to the 6 people, with red edges indicating the friendship relation, blue edges the stranger relation.

□

6-9. (Krausz's Theorem) The line graph L(G) of a linear graph G is defined as follows:

1. There is a one-to-one correspondence between the vertices in L(G) and the edges in G.
2. There is an edge joining two vertices in L(G) when their corresponding edges in G are incident with a common vertex.

Prove that a linear graph H without loops is a line graph of some linear graph if and only if there is a partition of the edges of H which is such that each subset in the partition forms a complete graph and every vertex in H is incident with edges that are in at most two subsets in the partition.

Let H be a linear graph without loops.

Assume there is a partition π of the edges of H into blocks which are complete subgraphs and every vertex of H is incident with edges in at most two blocks. Construct a graph G as follows. There is a vertex in G for each block in π. Two of these vertices in G are connected if the corresponding blocks of H have a vertex in common. For a vertex in H that is incident with edges in only one block π_x, there is an edge in G between v_x and an additional vertex of degree 1. For each isolated vertex of H there is an isolated edge of G. Since no two blocks of π have a common edge, each two blocks have at most one common vertex. Thus, each pair of vertices in G has at most one edge and G is linear. By construction, H = L(G).

Assume H = L(G), where G is a linear graph. For each vertex in G of degree $k \geq 2$ there is a complete subgraph in H with k vertices. H has no other edges. Further, because G is linear, no two such complete subgraphs in H have an edge in common. Also, since every edge in G is incident with 2 vertices, every vertex in H is incident with edges in at most two blocks.

□

6-10. Show that a vertex x is an articulation point of a connected graph if and only if there exist two vertices a and b such that every path joining a and b passes through x.

Assume x is an articulation point of a connected graph G. Removal of x and all edges incident with x leaves a subgraph of G which is unconnected. Choose vertices a and b from different connected components of this subgraph. Then, every path joining a and b passes through x.

Assume there exist 2 vertices a and b in a connected graph G, such that every path joining a and b passes through x. Removal of x and all edges incident with x leaves a subgraph of G which is unconnected. Therefore x is an articulation point of G.

□

6-11. Show that a directed graph that contains an Euler circuit is strongly connected.

Any two vertices are connected by a path along the Euler circuit.

□

6-12. Prove that a connected graph possesses an Euler circuit if and only if it can be decomposed as a set of elementary circuits that have no edges in common.

By Corollary 6-3-1, it is equivalent to prove that all vertices of a connected graph G are of even degree iff G can be decomposed into a set of edge-disjoint elementary circuits.

Assume all vertices of a connected graph G are of even degree. Removing any elementary circuit from G leaves a subgraph all of whose vertices are of even degree. Repeating this operation gives a set of edge-disjoint elementary circuits that include all edges of G.

Assume a connected graph G can be decomposed into a set of edge-disjoint elementary circuits. Composition produces a graph all of whose vertices are of even degree.

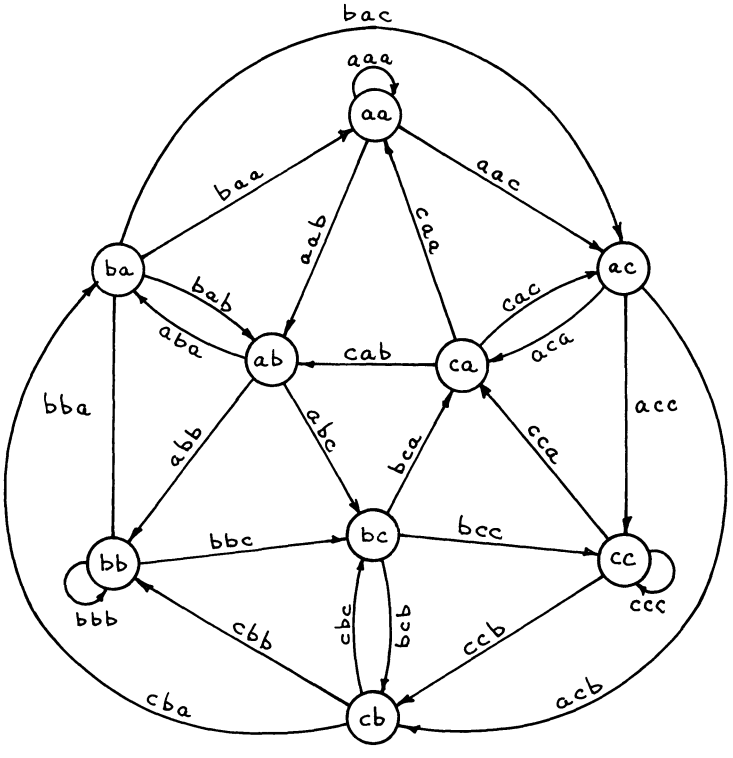
□

6-13. Prove that if a connected graph has k vertices of odd degree ($k > 0$), there are 1/2 k paths, no two of which have a common edge, that cover all the edges in the graph.

By Corollary 6-2-1, k is even. Add $\frac{k}{2}$ edges to the graph, connecting all odd-degree vertices in pairs. Find an Euler circuit in the resulting graph, and then delete the added edges. The required $\frac{k}{2}$ paths result.

□

6-14. Find a circular arrangement of nine a's, nine b's, and nine c's such that each of the 27 words of length 3 from the alphabet {a,b,c} appears exactly once.



Each vertex has incoming degree 3 and outgoing edgree 3. One possible circular arrangement corresponding to an Euler circuit is:

a a a b a b b b a a c a c c c a b c c b c b a c b b c

□

6-15. Is it possible to move a knight on an 8 × 8 chessboard so that it completes every possible move exactly once? A move between two squares of the chessboard is completed when it is made in either direction.

Construct an undirected graph G containing 64 vertices, one for each square of the chessboard. An edge connects two vertices iff a knight can make the corresponding move. The problem then is to find an Euler path for G. The degrees of the vertices are shown below on the corresponding chessboard square.

2	3	4	3	2
3	4	6	4	3
4	6	8	6	4
3	4	6	4	3
2	3	4	3	2

Since there are 8 vertices of degree 3, no Euler path exists.

□

6-16. Do there exist graphs in which an Euler path is also a Hamiltonian path? Characterize this class of graphs.

Yes.

Graphs which are just elementary paths.

□

6-17. n teams play in a round-robin tournament. Show that they can be ordered according to their winning records so that each team immediately precedes a team it has beaten. (The ordering is by no means unique.)

Construct a directed graph G having n vertices corresponding to the n teams, with

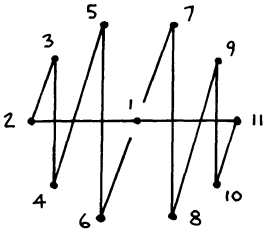
$(v_i, v_j) \in G$ iff team i has beaten team j.

G is a complete directed graph and, by Example 6-2, contains a Hamiltonian path, which gives the desired ordering.

□

6-18. Eleven students plan to have dinner together for several days. They will be seated at a round table, and the plan calls for each student to have different neighbors at every dinner. For how many days can this be done?

The problem is to find a maximum number of edge-disjoint Hamiltonian circuits in a complete graph with 11 vertices. There can be at most 5, since the graph contains $\binom{11}{2} = 55$ edges and each Hamiltonian circuit contains 11 edges. That there are 5 edge-disjoint Hamiltonian circuits can be seen as follows. The subgraph below is a Hamiltonian circuit.



Keeping the vertices fixed and rotating this pattern clockwise by $\frac{\pi}{5}$, $\frac{2\pi}{5}$, $\frac{3\pi}{5}$ and $\frac{4\pi}{5}$ gives 4 more Hamiltonian circuits, and no 2 of the 5 Hamiltonian circuits have a common edge.

□

6-19. Let G be a linear graph with n vertices. Show that G has a Hamiltonian path if the sum of the degrees of any two vertices is equal to or larger than n-1.

Hint: Show the existence of an elementary path of length n-1 by induction on the length of the path.

Let $\rho(v_i)$ denote the degree of vertex v_i.

G is connected. If G contained several connected components, then for v_1, v_2 taken from different connected components containing k_1 and k_2 vertices, because G is linear,

$$\rho(v_1) + \rho(v_2) \leq (k_1-1) + (k_2-1) = k_1 + k_2 - 2 \leq n-2$$

which is a contradiction.

Basis. There is an elementary path of length 1 in G, namely any edge $(v_1, v_2) \in G$.

Induction Hypothesis. Assume there is an elementary path of length ℓ in G, $\ell = 1,2,\cdots,n-2$.

Induction Step. Show there is an elementary path of length $\ell+1$ in G.

Let P be an elementary path of length ℓ in G.

Two cases are possible.

(i) v_1 or $v_{\ell+1}$ connects to one of the remaining vertices.

(ii) All edges incident with v_1 or $v_{\ell+1}$ terminate on vertices in P.

Case (i). Clearly P can be extended to length $\ell+1$.
Further, because

$$\rho(v_1) + \rho(v_{\ell+1}) \geq n-1$$

case (i) must hold for

$$\ell < \begin{cases} \frac{n}{2} & n \text{ - even} \\ \frac{n-1}{2} & n \text{ - odd} \end{cases}$$

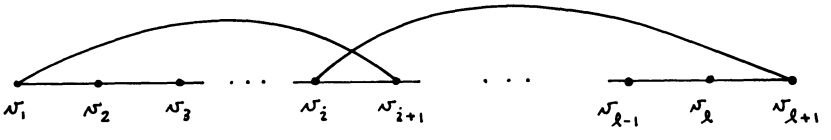
Case (ii). Let G' be the subgraph of G containing vertices $v_1, v_2, \cdots, v_{\ell+1}$ and all edges which connect 2 of these vertices. Show that G' has a Hamiltonian circuit, which leads to a way of extending P to length $\ell+1$.

If $(v_1, v_{\ell+1}) \in G'$, G' has a Hamiltonian circuit. Otherwise, since

$$\rho(v_1) + \rho(v_{\ell+1}) \geq \ell+1 \qquad \text{for } \ell = 1,2,\cdots, n-2$$

it will be shown that

$$(v_1, v_{i+1}) \text{ and } (v_i, v_{\ell+1}) \in G' \qquad \text{for some } i \in \{2,3,\cdots, \ell-1\}$$



and G' has the Hamiltonian circuit seen in the above figure by deleting edge (v_i, v_{i+1}).

To show this, let

$$\rho(v_{\ell+1}) = k \qquad k = 1,2,\cdots, \ell-1$$

(recall that $(v_1, v_{\ell+1}) \notin G'$)

and assume for each of the (k-1) edges

$$(v_i, v_{\ell+1}) \in G' \qquad i \in \{2,3,\cdots, \ell-1\}$$

($(v_\ell, v_{\ell+1}) \in G'$ is not used in this argument)

that the edge $(v_1, v_{i+1}) \notin G'$.

Then

$$\rho(v_1) \leq (\ell-1) - (k-1) = \ell-k$$

But

$$\rho(v_1) \geq (\ell+1) - \rho(v_{\ell+1}) = \ell-k+1$$

which is a contradiction.

Since G is connected, there must be an edge

$$(v_x, v_j) \in G \qquad v_x \text{ in } G - G'$$
$$v_j \text{ in } G'$$

and an elementary path of length $(\ell+1)$, using the edge (v_x, v_j) and the Hamiltonian circuit for G', less (v_j, v_{j+1}).

□

6-20. Show that a group of people can be seated around a table such that every one will have two of his friends at his two sides if every one knows at least half of the people in the group.

Hint: Extend the condition in Problem 19 to that for the existence of a Hamiltonian circuit in a graph.

A linear graph G with n vertices has a Hamiltonian circuit if the sum of the degrees of any two vertices is greater than or equal to n.

(Take the Hamiltonian path for G, shown to exist in Problem 6-19, and apply the method of the induction step, case (ii), with

$$\ell = n-1$$

$$\rho(v_1) + \rho(v_n) \geq n$$

to obtain a Hamiltonian circuit for G.)

Construct a graph G with n vertices, one for each person, in which there is an edge between two vertices iff the corresponding people are friends. For any two vertices v_i, v_j in G

$$\rho(v_i) \geq \frac{n}{2}$$

$$\rho(v_j) \geq \frac{n}{2}$$

$$\rho(v_i) + \rho(v_j) \geq n$$

and G has a Hamiltonian circuit, which gives the required seating plan.

□

7-1. Prove that the complement of a spanning tree does not contain a cut-set and that the complement of a cut-set does not contain a spanning tree.

A spanning tree has at least one edge in common with every cut-set, by Theorem 7-5. Therefore, the complement of a spanning tree lacks one or more edges from every cut-set.

A cut-set has at least one edge in common with every spanning tree, again by Theorem 7-5. Therefore, the complement of a cut-set lacks one or more edges of every spanning tree.

□

7-2. Let L be a circuit in a graph G. Let a and b be any two edges in L. Prove that there exists a cut-set C such that $L \cap C = \{a,b\}$.

Let T be a spanning tree in G containing all the edges L-a. Let C be the fundamental cut-set, relative to T, corresponding to the tree branch b.
Then,

$$\{L-a\} \cap C = b$$

and by Theorem 7-6,

$$L \cap C = \{a,b\}$$

□

7-3. Let T_1 and T_2 be two spanning trees of a connected graph G. Let a be an edge that is in T_1 but not T_2. Prove that there is an edge b in T_2 but not T_1 such that both $(T_1 - \{a\}) \cup \{b\}$ and $(T_2 - \{b\}) \cup \{a\}$ are spanning trees of G.

Let C_1 be the fundamental cut-set relative to T_1 corresponding to edge a.

$$C_1 \cap T_1 = \{a\} \qquad [1]$$

Since $a \notin T_2$, a is a chord of T_2 and defines a fundamental circuit L relative to T_2.

$$L \cap \overline{T}_2 = \{a\} \qquad [2]$$

Applying Theorem 7-6, $L \cap C_1$ contains an even number of edges, one of which is a. Let b be any other.

Since $b \in L$, by [2] $b \in T_2$.

Since $b \in C_1$, by [1] $b \notin T_1$, and $(T_1 - \{a\}) \cup \{b\}$ is a spanning tree.

Let C_2 be the fundamental cut-set relative to T_2 corresponding to edge b.

$$C_2 \cap T_2 = \{b\} \qquad [3]$$

Applying Theorem 7-6, $L \cap C_2$ contains an even number of edges, one of which is b. By [3], all others lie in $\overline{T}_2$. By [2], $L \cap C_2 = \{a,b\}$.

Since $a \in C_2$, $(T_2 - \{b\}) \cup \{a\}$ is a spanning tree.

□

7-4. (a) Let L_1 and L_2 be two circuits in a graph G. Let a be an edge that is in both L_1 and L_2, and let b be an edge that is in L_1 but not L_2. Prove that there exists a circuit L_3 which is such that $L_3 \subseteq (L_1 \cup L_2) - \{a\}$ and $b \in L_3$.

(b) Repeat part (a) when the term "circuit" is replaced by the term "cut-set".

(a) Let $L = L_1 \oplus L_2$

Then

$L \subseteq L_1 \cup L_2$

$a \notin L$, $b \in L$

or

$L \subseteq L_1 \cup L_2 - \{a\}$ and $b \in L$

By Theorem 7-9, L is either a circuit or edge-disjoint union of circuits. If L is a circuit, let $L_3 = L$. If L is an edge-disjoint union of circuits, let L_3 be the one circuit in L that contains the edge b.

(b) In the solution for part (a), replace "circuit" with "cut-set" and apply Theorem 7-11 in place of Theorem 7-9.

□

7-5. Let D be a set in edges in a graph G having the following two properties:

1. There is no cut-set C in G for which $D \cap C$ consists of a single edge.

2. For any two edges a and b in D, there is a cut-set C in G for which $D \cap C = \{a,b\}$.

Prove that D is a circuit or an edge-disjoint union of circuits.

First prove that there is no cut-set C in G for which $|D \cap C| = 2k+1$, $k = 0,1,2,\cdots$, by induction on k.

Basis. $k = 0$. Given by property (i).

Induction Hypothesis. Assume true for $k < n$.

Induction Step. Suppose there is a cut-set C in G for which $|D \cap C| = 2n+1$. By property (ii), for any 2 edges a and b in $D \cap C$ there is a cut-set C_1 in G for which $D \cap C_1 = \{a,b\}$. $C \oplus C_1$ is a cut-set or edge-disjoint union of cut-sets, by Theorem 7-11, and $D \cap (C \oplus C_1) = D \cap C - \{a,b\}$, so that $|D \cap (C \oplus C_1)| = 2n-1$. If $C \oplus C_1$ is a cut-set, this contradicts the induction hypothesis. If $C \oplus C_1$ is an edge-disjoint union of cut-sets, then at least one of these cut-sets has $2k+1$ edges, $k < n$, in common with D, again contradicting the induction hypothesis.

Now, since D has an even number of common edges with every cut-set in G, D has an even number of common edges with every vector in the cut-set subspace of G. By Theorem 7-17, D is a circuit or edge-disjoint union of circuits.

□

7-6. Are there graphs for which the cut-set subspace is the same as the vector space of all the subsets of edges? If so, characterize this class of graphs.

In such a graph, every edge must be a cut-set. Every subset of edges containing two or more edges is then an edge-disjoint union of cut-sets.

A graph meets this condition iff it contains no circuits. Each connected component of a graph containing no circuits is a tree. The class of graphs may be characterized as the class of forests.

□

7-7. Let G be a graph that has v vertices and v-1 edges. Prove that G is a tree if and only if any v-1 rows of the incidence matrix of G form a non-singular matrix. (Addition and multiplication are conducted in the field of integers modulo 2.)

With v vertices and (v-1) edges, G is a tree iff G is connected.

If G is connected, then:

The dimension of the cut-set subspace is (v-1).

Any (v-1) rows of the incidence matrix are linearly independent. (Using Theorem 7-20).

Any (v-1) rows of the incidence matrix form a non-singular matrix.

If any (v-1) rows of the incidence matrix form a non-singular matrix, then

Any (v-1) rows of the incidence matrix are linearly independent.

The dimension of the cut-set subspace is (v-1).

G is connected.

□

7-8. Show that there exists no graph having a circuit matrix equal to M_A. Show also that there exists no graph having a cut-set matrix equal to M_B.

$$M_A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad M_B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Assume M_A is the circuit matrix for some graph G.

$$M_A = \left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

fundamental system of circuits

branches chords

The dimension of M_A is (e-v+1)x e, e = 7, v = 4. Since $\binom{4}{2} = 6$ and e = 7, G is not a linear graph, and must contain at least one circuit of length 2. But no linear combination of the rows of M_A can yield a circuit with just 2 edges, hence M_A is not the circuit matrix for any graph G.

Assume M_B is the cut-set matrix for some graph G.

$$M_B = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right]$$

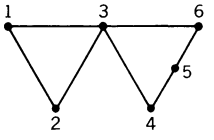
fundamental system of cut-sets.

branches chords

As above, G is not a linear graph and must contain at least one circuit of length 2. The 2 edges of such a circuit must either participate jointly in a cut-set or be jointly absent. The corresponding 2 columns or M_B must be identical. Since no 2 columns of M_B are identical M_B is not the cut-set matrix for any graph G.

□

7-9. The adjacency matrix P of a graph with v vertices is a v × v matrix, the (i,j)th entry of which is 1 if there is an edge joining the ith and the jth vertices, and is 0



(a)

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

(b)

Figure 7P-1

otherwise. For example, the adjacency matrix of the graph in Fig. 7P-1a is shown in Fig. 7P-1b.

(a) Define the addition and multiplication operations for the elements 0 and 1 such that when P^k is computed, a 1 in the (i,j)th entry of P^k means that there is a path of length k between the ith and the jth vertices. Check your result by computing P^2 for the matrix in Fig. 7P-1b.

(b) Instead of the 0's in the diagonal, we shall fill the diagonal of P with 1's. For the addition and multiplication operations defined in part (a), what are the significances of a 0 entry and of a 1 entry in P^k?

(a) Use the addition and multiplication operations of the field modulo 2.

(b)
$$P^k(i,j) = \begin{cases} 1 & \text{if there is a path of length} \leq k \text{ between the } i^{th} \text{ and } j^{th} \text{ vertices} \\ 0 & \text{otherwise} \end{cases}$$

□

7-10. Let G be a biconnected graph, and let a and b be two distinct vertices in G. Show that there exists a basis of the cut-set subspace which is such that every cut-set in this basis places a and b in two different connected components.

Let G contain k vertices.

Denote by $\underline{v}_i$ the i^{th} row of the incidence matrix for G, for some arbitrary ordering of vertices in G. $\underline{v}_i$ is a cut-set (or, in general, an edge-disjoint union of cut-sets in case

vertex v_i is an articulation point). By Theorem 7-20, $V = \{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_{k-1}\}$ is a basis for the cut-set subspace. Define a new set of (k-1) vectors $C = \{\underline{c}_1, \underline{c}_2, \cdots, \underline{c}_{k-1}\}$ as follows:

$$\begin{aligned}
\underline{c}_1 &= \underline{v}_1 \\
\underline{c}_2 &= \underline{v}_1 \oplus \underline{v}_2 \\
c_3 &= \underline{v}_1 \oplus \underline{v}_2 \oplus \underline{v}_3 \\
&\vdots \\
\underline{c}_{k-1} &= \underline{v}_1 \oplus \underline{v}_2 \oplus \underline{v}_3 \oplus \cdots \oplus \underline{v}_{k-1}
\end{aligned}$$

Each of these is a cut-set or edge-disjoint union of cut-sets, by Corollary 7-11-1. Further, C is a basis for the cut-set subspace since each vector in the basis V can be expressed as a linear combination of vectors in C as follows:

$$\begin{aligned}
\underline{v}_1 &= \underline{c}_1 \\
\underline{v}_j &= \underline{c}_j \oplus \underline{c}_{j-1} \qquad j = 2,3,\cdots, k-1
\end{aligned}$$

C is the required basis, once the rows of the incidence matrix are ordered in the following non-unique way, to insure:

(i) Each $\underline{c}_j \in C$ separates vertices a and b

(ii) C contains only cut-sets.

To satisfy (i), set $v_1 = a$, $v_k = b$.

It remains to satisfy (ii).

$\underline{c}_1 = \underline{v}_1$ is a cut-set, since G contains no articulation points.

Assume that ℓ vertices $v_1, v_2, \cdots, v_\ell$ have been ordered such that $\underline{c}_1, \underline{c}_2, \cdots, \underline{c}_\ell$ are cut-sets. The $(\ell+1)^{\underline{st}}$ vertex is selected in the following manner.

$\underline{c}_\ell$ specifies the set of edges between two connected subgraphs X and Y, where X consists of the vertices $v_1, v_2, \cdots, v_\ell$ and edges connecting these, and Y consists of the remaining vertices, and edges connecting these.

Consider the vertices in Y adjacent to vertices in X. If one of these, say v_x, is not an articulation point in subgraph Y, then let $v_{\ell+1} = v_x$. $\underline{c}_{\ell+1}$ is then a cut-set, since by moving v_x from Y to X, X remains connected (v_x is adjacent to one or more vertices in X) and Y remains connected (v_x is not an articulation point in Y).

Suppose no such vertex v_x exists in Y. That is, every vertex in Y adjacent to vertices in X is an articulation point in Y. It is shown that one of these must be an articulation point in G, contradicting the fact that G is bi-connected.

Divide Y into pieces (connected components) by splitting all its articulation points. Each piece contains one or more articulation points of Y. There must be a piece containing exactly one articulation point v_y of Y, otherwise there would be an elementary circuit in Y passing thru two or more pieces of Y, which is impossible. Then v_y is an articulation point of G, since removal of v_y and all incident edges in G leaves Y with two or more connected components, one of which contains no vertices adjacent to vertices in X.

□

7-11. (Whitney's Theorem) Two graphs G and G' are said to be edge isomorphic when there is a one-to-one correspondence between the sets of edges in G and G' such that e_1 and e_2 in G are adjacent (incident with a common vertex) if and only if the corresponding edges e_1' and e_2' in G' are also adjacent. Note that the fact that G and G' are edge isomorphic does not necessarily imply that they are isomorphic. The two graphs shown in Fig. 7P-2 are edge isomorphic, but not isomorphic. However, if G and G' are isomorphic, then clearly they are edge isomorphic.

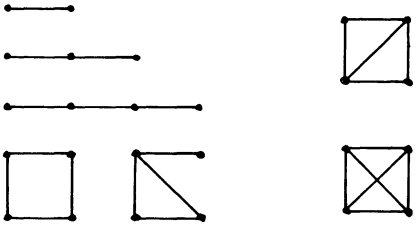
Figure 7P-2

Whitney's theorem implies that any graph which is edge isomorphic to a linear connected graph G, other than the two graphs shown above, is isomorphic to G.

(a) Show that the theorem holds for all non-isomorphic linear connected graphs having four or fewer vertices, other than the two shown above.

(b) Prove that for any linear connected graph G having five or more vertices, a graph which is edge isomorphic to G is isomorphic to G and the edge isomorphism is that specified by the isomorphism. Use induction on the number of edges in G, with trees containing five vertices as a basis.

(a) With G taken as any one of the following



it may be seen that any graph edge isomorphic to G is isomorphic to G.

(b) The proof is that given by Ore in Reference 5.

Basis. Let G be a tree containing 5 vertices, namely one of the following three graphs.

It may be seen that any graph edge isomorphic to G is isomorphic to G, and the edge isomorphism is that specified by the isomorphism.

Induction Hypothesis. Assume the theorem is true for any linear connected graph G having m edges and 5 or more vertices.

Induction Step. Add a single edge e to G to obtain a linear connected graph H. Consider a graph H' edge isomorphic to H. Let G' be the set of edges in H' corresponding to those in G, and e' the edge in H' corresponding to e. G' is edge isomorphic to G. By the induction hypothesis, G' is isomorphic to G, and the isomorphism specifies the edge isomorphism. Thus, H' consists of G and an added edge e'.

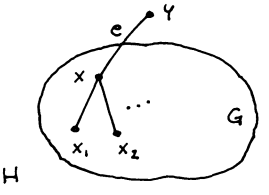
Let $e = \{x,y\}$, $e' = \{x',y'\}$

Let $X = \{\{x,x_1\}, \{x,x_2\}, \cdots\}$, the set of edges incident with vertex x in G.

Let $Y = \{\{y,y_1\}, \{y_1,y_2\}, \cdots\}$, the set of edges incident with vertex y in G.

Case (i).

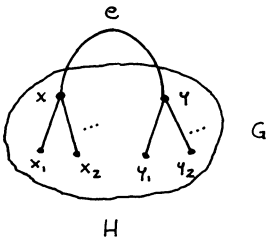
In H, x is in G, y is not. e is adjacent only to edges X.



In H', x' and y' cannot both be vertices in G, otherwise e would be adjacent to edges other than X. x' and y' cannot both lie outside G, otherwise H' would not be connected. Thus, x' is in G and y' is not (or vice versa). Further, x' = x, otherwise e' would be adjacent to edges other than X.

Case (ii).

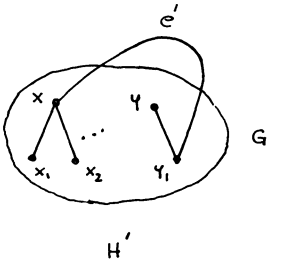
In H, x and y are both in G. e is adjacent to edges X and Y.



In H',

(ii)(a) Assume $x' = x$, $y' \neq y$ (or $x' \neq y$, $y' = y$).

Since e' must be adjacent to edges Y, there can be just one edge $\{y,y_1\}$ in Y, with $y' = y_1$

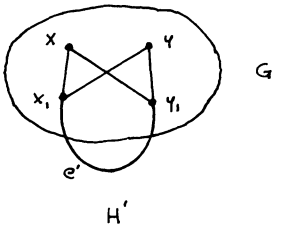


But then e' is adjacent to additional edges incident with y_1, otherwise edge $\{y,y_1\}$ is isolated in G.

(ii)(b) Assume $x' \neq x$, $y' \neq y$.

Since e' must be adjacent to edges X, there can be at most 2 edges $\{x,x_1\}$ and $\{x,x_2\}$ in X, with $e' = \{x_1,x_2\}$.

Also, there can be at most 2 edges in Y, $\{y,x_1\}$ and $\{y,x_2\}$.



But then e' is adjacent to other edges at x_1 or x_2, since G is connected and contains 5 or more vertices.

Therefore, $x' = x$ and $y' = y$.

The conclusion is that H' is isomorphic to H, and that isomorphism specifies the edge isomorphism.

□

7-12. In directed graphs, an oriented circuit is defined as a set of edges that is a circuit in the undirected graph obtained from the directed graph by ignoring the direction of the edges. Similarly, an oriented cut-set is a set of edges that is a cut-set in the undirected graph obtained in the same manner. The orientation of an oriented circuit is specified by a cyclic ordering of the vertices it encounters, that of an oriented cut-set by an ordering of the two subsets of vertices it separates. An oriented circuit vector or an oriented cut-set vector in a directed graph with e edges is represented by an ordered e-tuple. The jth component of an oriented circuit vector $\vec{L}$ is either 1, -1, or 0 accordingly as the jth edge is in the oriented circuit and its direction agrees with the circuit orientation, the jth edge is in the oriented circuit and its direction is opposite to the circuit orientation, or the jth edge is not in the oriented circuit. Similarly, for an oriented cut-set vector $\vec{C}$. These vectors are considered as vectors in an e-dimensional vector space over the field of real numbers.

(a) Let $\vec{L}$ and $\vec{C}$ be an oriented circuit vector and an oriented cut-set vector, respectively, in a directed graph. Show that $\vec{L} \cdot \vec{C} = 0$.

(b) Prove that any oriented circuit vector $\vec{L}$ can be expressed as a linear combination of fundamental oriented circuit vectors.

(c) Prove that any oriented cut-set vector $\vec{C}$ can be expressed as a linear combination of fundamental oriented cut-set vectors.
(The orientation of a fundamental oriented circuit is taken to agree with that of its defining chord. Similarly, for a fundamental oriented cut-set and its defining branch.)

(a) Ignoring the directions of edges, $\vec{L}$ has an even number of edges in common with $\vec{C}$, by Theorem 7-6.

Assume first that all of these common edges agree in orientation with $\vec{L}$, i.e., the corresponding components or $\vec{L}$ are all 1. Then, since $\vec{L}$ traverses the cut-set $\vec{C}$ the same number of times in each direction, $\vec{L} \cdot \vec{C} = 0$.

If not all common edges agree in orientation with $\vec{L}$, then the situation differs from the one assumed above only in that some of the common non-zero components of $\vec{L}$ and $\vec{C}$ are both negated, and $\vec{L} \cdot \vec{C}$ remains 0.

(b) Let $\vec{L}$ be any circuit vector of a directed graph. Let $\vec{L}$ contain the chords $\{e_1, e_2, \cdots, e_k\}$ relative to some spanning tree, and no others. Let $\vec{L}_1$ be the vector obtained by summing the k fundamental circuit vectors defined by these chords, after negating the fundamental circuit vector defined by e_i, $i = 1,2,\cdots,k$, if the orientation of e_i is opposite that of $\vec{L}$.

$\vec{L}_1$ contains the chords $\{e_1, e_2, \cdots, e_k\}$ and no others. The corresponding components of $\vec{L}_1$ agree in sign with those of $\vec{L}$. If $\vec{L} - \vec{L}_1$ has non-zero components, they must correspond only to branches.

Consider the fundamental cut-set vector $\vec{C}$ defined by any tree branch e_b. By part (a), $\vec{L} \cdot \vec{C} = 0$. Also, since $\vec{L}_1$ is a linear combination of circuit vectors, $\vec{L}_1 \cdot \vec{C} = 0$. Thus $(\vec{L} - \vec{L}_1) \cdot \vec{C} = 0$. But since e_b is the only tree branch in $\vec{C}$, the component of $\vec{L} - \vec{L}_1$ corresponding to e_b must be 0.

Therefore, $\vec{L} - \vec{L}_1$ is the zero vector and $\vec{L} = \vec{L}_1$.

(c) The proof is the dual of that of part (b).

Let $\vec{C}$ be any cut-set vector of a directed graph. Let $\vec{C}$ contain the branches $\{e_1, e_2, \cdots, e_k\}$ relative to some spanning tree, and no others. Let $\vec{C}_1$ be the vector obtained by summing the k fundamental cut-set vectors defined by these branches, after negating the fundamental cut-set vector defined by e_i, $i = 1,2,\cdots,k$, if the orientation of e_i is opposite that of $\vec{C}$.

$\vec{C}_1$ contains the branches $\{e_1, e_2, \cdots, e_k\}$ and no others. The corresponding components of $\vec{C}_1$ agree in sign with those of $\vec{C}$. If $\vec{C} - \vec{C}_1$ has non-zero components, they must correspond only to chords.

Consider the fundamental circuit vector $\vec{L}$ defined by any chord e_c. By part (a), $\vec{L} \cdot \vec{C} = 0$. Also, since $\vec{C}_1$ is a linear combination of cut-set vectors, $\vec{L} \cdot \vec{C}_1 = 0$. Thus, $\vec{L} \cdot (\vec{C} - \vec{C}_1) = 0$. But since e_c is the only chord in L, the component of $\vec{C} - \vec{C}_1$ corresponding to e_c must be 0.

Therefore, $\vec{C} - \vec{C}_1$ is the zero vector and $\vec{C} = \vec{C}_1$.

8-1. Prove that in a connected planar graph in which every vertex is of at least degree 3, there exists a region with fewer than six edges in the boundary.

Let G be such a graph.

By Theorem 8-7, G has a dual G^d.

G^d is planar, by Corollary 8-6-2 and Theorem 8-7.

G^d is linear, since G contains no vertices of degree 2.

Using the result of Example 8-4, G^d contains a vertex of degree ≤ 5.

Corresponding to this vertex in G^d is a contour in G containing ≤ 5 edges.

□

8-2. Show that in a planar graph with six vertices and twelve edges, each of the regions is bounded by three edges.

Using Euler's formula, the number of regions is

$$r = e-v+2 = 8$$

Count the edges in each contour, and sum over all contours to obtain a total T.

Since the graph is linear, each contour contains at least 3 edges and

$$T \geq 3r$$

Furthermore, each edge lies in at most 2 contours, so

$$T \leq 2e$$

$$3r \leq T \leq 2e$$

In this case,

$$3 \cdot 8 = T = 2 \cdot 12$$

Since there are 8 regions, each bounded by at least 3 edges, each must be bounded by exactly 3 edges.

□

8-3. In this problem, Euler's formula is derived in an alternative way.

(a) According to Theorem 8-2, a planar graph can be mapped on the plane in such a way that the edges are all straight-line segments. It follows that the regions are all polygons. Let $n_1, n_2, \cdots, n_{r-1}$ denote the numbers of edges in the boundaries of the r-1 finite regions. Show that the sum of the interior angles of these r-1 polygons, T, is equal to

$$[n_1 + n_2 + \cdots + n_{r-1} - 2(r-1)] \cdot 180^\circ$$

(b) Let e denote the number of edges in the graph, and let e_x denote the number of edges in the boundary of the infinite region. Show that

$$T = [2e - e_x - 2(r-1)] \cdot 180^\circ$$

(c) The quantity T can be computed in another way. Let v denote the number of vertices in the graph, and let v_x denote the number of vertices in the boundary of the infinite region. Show that

$$T = (v - v_x) \cdot 360^\circ + (e_x - 2) \cdot 180^\circ$$

(d) Using $e_x = v_x$, obtain Euler's formula by equating the results in parts (b) and (c)

(a) The sum of the interior angles of an n-gon is $(n-2)\cdot 180^\circ$. This follows from the fact that the interior of an n-gon may be divided into (n-2) non-overlapping triangles, the sum of whose interior angles is the desired result.

$$T = (n_1-2) \cdot 180^\circ + (n_2-2) \cdot 180^\circ + \cdots + (n_{r-1}-2) \cdot 180^\circ$$
$$= [n_1 + n_2 + \cdots + n_{r-1} - 2(r-1)] \cdot 180^\circ \qquad [1]$$

(b) Each of the e_x edges in the boundary of the infinite region is in the boundary of exactly 1 finite region.

Each of the remaining $e-e_x$ edges is in the boundary of exactly 2 finite regions.

$$\sum_{i=1}^{r-1} n_i = e_x + 2(e-e_x) = 2e-e_x$$

Substituting in [1],

$$T = [2e-e_x - 2(r-1)] \cdot 180^\circ \qquad [2]$$

(c) Each of the $(v-v_x)$ vertices not in the boundary of the infinite region contributes 360° to T.

The sum of interior angles of the e_x-gon bounding the infinite region is $(e_x-2) \cdot 180^o$.

$$T = (v-v_x) \cdot 360^o + (e_x-2) \cdot 180^o \qquad [3]$$

(d) Equating [2] and [3]

$$2e-e_x - 2(r-1) = 2(v-v_x) + e_x-2$$

and with $e_x = v_x$

$$2e-e_x - 2(r-1) = 2v - e_x-2$$

$$v-e+r = 2$$

□

8-4. Prove that in a connected planar graph the ring sum of the contours of the finite regions is the contour of the infinite region.

Each edge in the contour of the infinite region appears in the contour of exactly 1 finite region.

Every other edge appears in the contour of exactly 2 finite regions.

□

8-5. Prove that if G and $\overline{G}$ are two complementary graphs with 11 or more vertices, then either G or $\overline{G}$ is non-planar. (As a matter of fact, the statement is true for G and $\overline{G}$ being complementary graphs with nine or more vertices. However, the proof is much harder.)

Let $v = 11$.

The complete graph contains $\binom{11}{2} = 55$ edges.

G or $\overline{G}$ contains 28 or more edges.

Assume both G and $\overline{G}$ are planar.

From Example 8-1, for any planar graph

$$e \leq 3v-6$$

In this case

$$e \leq 27$$

Thus, G or $\overline{G}$ is non-planar.

For $v > 11$, any pair of complementary graphs have as subgraphs a pair of complementary graphs with 11 vertices, one of which must be non-planar.

□

8-6. Let G be a planar graph the degree of each vertex of which is 3. Prove that a dual of G will have an odd number of finite regions.

By Corollary 6-2-1, G contains an even number of vertices. Since there is one region in a dual for each vertex in G, a dual will have an even number of regions, and an odd number of finite regions.

□

8-7. Find two non-isomorphic dual graphs of the graph in Fig. 8P-1.

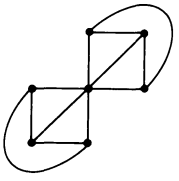


Fig. 8P-1

One dual is the graph itself.

Another dual, 2-isomorphic to the first, is obtained by separating the graph into two connected components by splitting its articulation point.

□

8-8. Find two non-isomorphic dual graphs of the graph in Fig. 8P-2.

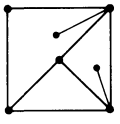
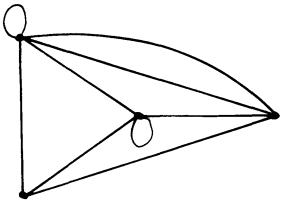
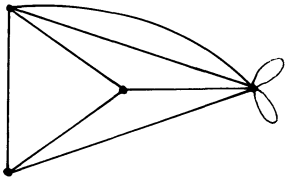


Fig. 8P-2





□

8-9. Show that a dual of a planar biconnected graph containing no loops is also biconnected.

Let G_1 be a loop-free, planar, biconnected graph and G_2 a dual of G_1.

Assume that G_2 contains an articulation point.

Divide G_2 into two or more connected pieces by splitting the articulation point to obtain G_2'. G_2' is 2-isomorphic to G_2 and, by Theorem 8-8, G_2' is also a dual of G_1.

By Corollary 8-6-2, G_1 is a dual of G_2'. In G_2', each piece contains one or more finite regions, otherwise G_1 would contain loops. Let a and b be two vertices in G_1 corresponding to finite regions in different pieces of G_2'. Let x be the vertex in G_1 corresponding to the infinite region in G_2'. Every path joining a and b in G_1 passes through x. Hence x is an articulation point of G_1, by Problem 6-10, which is a contradiction.

Therefore, G_2 is biconnected.

□

8-10. A series-parallel graph is a one-terminal-pair graph defined recursively as follows:

1. A single edge is a series-parallel graph.
   If G' and G'' are series-parallel then:
2. The series combination of G' and G'' is a series-parallel graph. By the series combination of G' and G'', we mean the joining of one of the terminals of G' with one of the terminals of G'' as illustrated in Fig. 8P-3a.
3. The parallel combination of G' and G'' is a series-parallel graph. By the parallel combination of G' and G'', we mean the joining of the two terminals of G' with the two terminals of G'' as illustrated in Fig. 8P-3b.

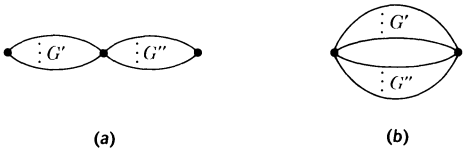


Fig. 8P-3

Show that a dual of a graph G is a series-parallel graph if and only if G is a series-parallel.

It is necessary only to show that if G is series-parallel then a dual G^d of G is series-parallel. Implication in the opposite direction follows from Corollary 8-6-2. The proof is by induction on the number of edges in G.

Basis. G consists of a single edge. G^d also consists of a single edge and is series-parallel by (1) above.

Induction Hypothesis. Assume that for any series-parallel graph containing m or fewer edges, a dual G^d is series-parallel.

Induction Step. Let G be a series-parallel graph containing (m+1) edges. In constructing G in accordance with the recursive definition above, the last operation consists of combining two series-parallel subgraphs G' and G'' containing m or fewer edges in series or in parallel.

(a) G' and G'' are combined in parallel to form G.

A dual G^d of G consists of a series combination of G'^d and G''^d, duals of G' and G'' respectively. G'^d and G''^d are series-parallel by induction hypothesis, and G^d is series-parallel by (2) above.

(b) G' and G'' are combined in series to form G. A dual G^d of G consists of a parallel combination of G'^d and G''^d. G^d is series-parallel by (3) above.

□

8-11. Let G_1 and G_2 be two graphs with a one-to-one correspondence between their edges. Let H_1 be any subgraph of G_1. Let n_1 denote the dimension of the circuit subspace of H_1. Let $\overline{H}_2$ be the complement of the corresponding subgraph of H_1 in G_2. Prove that

$$r_2 = R_2 - n_1$$

if and only if G_1 and G_2 are dual graphs, where r_2 and R_2 are the dimensions of the cut-set subspaces of $\overline{H}_2$ and G_2, respectively.

Assume G_1 and G_2 are dual graphs.

By the definition of duality, there is a one-to-one correspondence between the edges of G_1 and G_2 which establishes a one-to-one correspondence (moreover, an isomorphism) between

(i) The circuit subspace of G_1 and the cut-set subspace of G_2.

(ii) The cut-set subspace of G_1 and the circuit subspace of G_2.

Choose a spanning tree (or forest) for the subgraph H_1 and add edges from $\overline{H}_1$, if necessary, to obtain a spanning tree (or forest) for the graph G_1.

Consider the complement $\overline{T}_1$ of T_1 in G_1. There are R_2 edges in $\overline{T}_1$, n_1 of which belong to the subgraph H_1. The remaining R_2-n_1 edges in $\overline{T}_1$ belong to the subgraph $\overline{H}_1$.

$\overline{T}_1$ contains no cut-sets (see Problem 7-1). It follows that the corresponding set of edges $\overline{T}_2$ in G_2 contains no circuits. In particular, the R_2-n_1 edges of $\overline{T}_2$ that belong to $\overline{H}_2$ contain no circuits, and must therefore form a subgraph of a spanning tree (or forest) for $\overline{H}_2$.

Thus,

$$r_2 \geq R_2-n_1$$

or

$$r_2 = R_2-n_1+k \qquad k \geq 0 \qquad [1]$$

Similarly, interchanging the roles of G_1 and G_2, H_1 and $\overline{H}_2$, gives

$$r_1 \geq R_1-n_2$$

or

$$r_1 = R_1-n_2+k' \qquad k' \geq 0 \qquad [2]$$

where r_1, R_1 and n_2 are the dimensions of the cut-set subspace of H_1, the cut-set subspace of G_1 and the circuit subspace of $\overline{H}_2$, respectively.

Adding [1] and [2],

$$k+k' = (r_1+n_1) + (r_2+n_2) - (R_1+R_2)$$

Let e(G) denote the number of edges in a graph G. Then

$$r_1+n_1 = e(H_1) = e(H_2)$$

$$r_2+n_2 = e(\overline{H}_2) = e(\overline{H}_1)$$

$$R_1+R_2 = e(G_1) = e(G_2)$$

and

$$\begin{aligned} k+k' &= e(H_1) + e(\overline{H}_1) - e(G_1) \\ &= e(G_1) - e(G_1) \\ &= 0 \end{aligned}$$

Thus

$$k = k' = 0$$

and from [1],

$$r_2 = R_2 - n_1$$

Conversely, assume that

$$r_2 = R_2-n_1$$

for all subgraphs H_1 of G_1.

(i) If H_1 is a circuit in G_1, then H_2 is a cut-set in G_2. Take any circuit in G_1 as the subgraph H_1.

Then $n_1 = 1$.

Furthermore, if any edges are deleted from H_1, then $n_1 = 0$.

Now,

$$r_2 = \begin{cases} R_2 - 1 & \text{if } n_1 = 1 \\ R_2 & \text{if } n_1 = 0 \end{cases}$$

That is, the dimension of the cut-set subspace of G_2 is reduced by 1 by deleting the subgraph H_2 from G_2, and remains unchanged if any proper subgraph of H_2 is deleted from G_2. Also,

$$R_2 = v_2 - p_2$$

where v_2 and p_2 are the number of vertices and connected components in G_2, respectively

Removing the subgraph H_2 from G_2 cannot increase v_2 nor decrease p_2. It must therefore either decrease v_2 by 1 or increase p_2 by 1 (not both). The removal of any proper subgraph of H_2 from G_2 leaves v_2 and p_2 unchanged.

H_2 is therefore a cut-set in G_2.

(ii) If H_1 is an edge-disjoint union of circuits in G_1, then H_2 is an edge-disjoint union of cut-sets in G_2.

Take any edge-disjoint union of circuits in G_1 as the subgraph H_1.

To each circuit of H_1 there corresponds a cut-set in G_2, as shown in (i) above. The union H_2 of these cut-sets is an edge-disjoint union of cut-sets in G_2.

The converses to (i) and (ii) are proven similarly. Therefore, G_1 and G_2 are dual graphs.

□

9-1. Let G be a graph with n vertices. Show that $\beta(G)\gamma(G) \geq n$.

Let G be properly colored with $\gamma(G)$ distinct colors. Partition the n vertices of G into $\gamma(G)$ blocks

$$B_1, B_2, \cdots, B_{\gamma(G)}$$

where two vertices are in the same block iff they are of the same color.

Since each block is an independent set,

$$|B_i| \leq \beta(G) \qquad i = 1,2,\cdots,\gamma(G).$$

and with

$$|B_1| + |B_2| + \cdots + |B_{\gamma(G)}| = n$$

it follows that

$$\underbrace{\beta(G) + \beta(G) + \cdots + \beta(G)}_{\gamma(G) \text{ terms}} \geq n$$

$$\beta(G) \cdot \gamma(G) \geq n$$

□

9-2. For any two graphs G and H, show that $\gamma(G \times H) \leq \gamma(G)\gamma(H)$.

Let G and H be properly colored with $\gamma(G)$ and $\gamma(H)$ colors respectively.

Let c(a) denote the color of vertex a.

Color a vertex (a,b) of $G \times H$ with the color (c(a), c(b)). $\gamma(G) \cdot \gamma(H)$ different colors will be needed to color all vertices of $G \times H$.

No two adjacent vertices in $G \times H$ receive the same color. For any vertex (a,b) of $G \times H$, adjacent vertices fall into three categories

(i) (a_i, b) where a and a_i are adjacent in G

since $c(a) \neq c(a_i)$

$(c(a), c(b)) \neq (c(a_i), c(b))$

(ii) (a, b_i) where b and b_i are adjacent in H

since $c(b) \neq c(b_i)$

$(c(a), c(b)) \neq (c(a), c(b_i))$

(iii) (a_i, b_i) where a and a_i are adjacent in G

and b and b_i " " " H.

since $c(a) \neq c(a_i)$ and $c(b) \neq c(b_i)$

$(c(a), c(b)) \neq (c(a_i), c(b_i))$

$G \times H$ is properly colored with $\gamma(G) \cdot \gamma(H)$ colors, hence

$$\gamma(G \times H) \leq \gamma(G) \cdot \gamma(H)$$

□

9-3. For any two graphs G and H, show that $\beta(G \times H) \geq \beta(G)\ \beta(H)$.

Let $I_G = \{a_1, a_2, \cdots, a_{\beta(G)}\}$ be an independent set in G

and $I_H = \{b_1, b_2, \cdots, b_{\beta(H)}\}$ " " " " " H.

The $\beta(G) \cdot \beta(H)$ vertices in $I_G \times I_H$ are an independent set in $G \times H$, hence

$$\beta(G \times H) \geq \beta(G) \cdot \beta(H)$$

□

9-4. Show that the regions of a planar graph cannot be properly colored with two colors if the graph has 9 vertices and 17 edges and the degree of each vertex is at least 3.

Let G be such a graph.

Assume G can be properly two colored.

By Corollary 9-2-1, all vertices in G have even degree.

Then all vertices have even degree ≥ 4.

Summing degrees over all vertices gives

$2e \geq 4v$

$34 \geq 36$, a contradiction.

Therefore G cannot be properly two colored.

□

9-5. Show that a linear planar graph with 17 edges and 10 vertices cannot be properly colored with two colors.

Let G be such a graph.

From Euler's formula, G has 9 regions. The contour of each region contains 3 or more edges because G is linear.

The problem of properly coloring the vertices of G is equivalent to the problem of properly coloring the regions of a dual of G.

A dual contains 9 vertices and 17 edges. The degree of each vertex is 3 or more. It was shown in Problem 9-4 that the regions of such a graph cannot be properly two colored.

□

9-6. Show that two colors are sufficient to color the regions into which the plane is divided by infinite straight lines drawn in an arbitrary manner.

Map the figure from the plane onto the surface of a sphere using a stereographic projection. The result is a planar graph whose vertices are the images of all intersections (if any) and the convergence at the north pole of the images of all the infinite lines. The degree of each vertex is even and, by Corollary 9-2-1, the regions of this graph can be properly two colored. Any such two coloring may be mapped back onto the plane to obtain the desired result.

□

9-7. A knight has made n moves on an 8 x 8 chessboard and has returned to the square from which it started. Prove that n must be an even number.

Construct an undirected graph G having 64 vertices, one corresponding to each square of the chessboard. There is an edge between two vertices iff the knight can make the corresponding move.

It is characteristic of the knight's move that the color of the destination square is different than the color of the starting square. It follows that assigning the colors of the chessboard squares to the corresponding vertices of G properly two colors G.

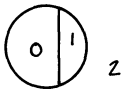
By Theorem 9-2, G contains no circuits of odd length. The number of moves made by the knight traversing any circuit in G must be even.

□

9-8 n circles are drawn on the plane. In each circle a chord is drawn so that chords in two different circles have at most one point in common. Show the regions into which the plane is divided can be properly colored with three colors.

HINT: Prove this by induction on the number of circles.

Basis. With one circle and its chord, the regions of the plane can be properly 3-colored as follows:



Induction Hypothesis. Assume that the regions of the plane formed by n circles and their chords, no two chords having more than one point in common, has been properly 3-colored.

Induction Step. Add an $(n+1)^{st}$ circle C in an arbitrary manner. C subdivides some previously existing regions into two regions, one internal to C, the other external to C. Permute the colors of all regions internal to C in the following way.

At this point, the regions are properly 3-colored since

(i) The colors of any two adjacent regions both external to C remain unchanged and different.

(ii) The colors of any two adjacent regions both internal to C have been permuted but remain different.

(iii) Any pair of adjacent regions, one internal to C, the other external to C, together form a previously existing region. The permutation of colors of regions internal to C insures that any such pair of regions are now colored differently.

Next, add a chord of C so that it has at most one point in common with any other chord. The new chord subdivides some regions internal to C into two regions. Permute the colors of all regions internal to C and lying on one side of the new chord in the same way as before. An extension of the above argument shows that the regions are properly 3-colored.

□

9-9. (a) Show that a linear planar graph with less than 30 edges has a vertex of degree 4 or less.

(b) Show that a linear planar graph with less than 30 edges can be properly colored with four colors.

(a) Since the graph is linear, each region is bounded by at least 3 edges and

$$2e \geq 3r$$

Assume the degree of each vertex is ≥ 5. Summing degrees over all vertices gives

$$2e \geq 5v$$

Using these two inequalities in Euler's formula

$$e = r + v - 2$$

$$\leq \frac{2e}{3} + \frac{2e}{5} - 2$$

$$15e \leq 10e + 6e - 30$$

$$e \geq 30$$

which is a contradiction. Therefore the graph must have a vertex of degree 4 or less.

(b) The proof is analogous to the proof of Theorem 9-3, with the result of part (a) replacing the result of Example 8-4.

□

9-10. (a) Show that if a planar graph has less than 12 regions and the degree of each vertex is at least 3, then there is a region bounded by four or fewer edges.

(b) Prove that four colors are sufficient to color the regions of such a graph.

(a) Summing degrees over all vertices gives

$$2e \geq 3v$$

Assume every region is bounded by 5 or more edges.
Then

$$2e \geq 5r$$

Using these two inequalities in Euler's formula

$$r = e - v + 2$$

$$\geq e - \frac{2e}{3} + 2 = \frac{e}{3} + 2 \geq \frac{5r}{6} + 2$$

$$6r \geq 5r + 12$$

$$r \geq 12$$

which is a contradiction. Therefore the map must contain a region bounded by 4 or fewer edges.

(b) The problem of properly coloring the regions of such a map is equivalent to the problem of properly coloring the vertices of a dual. By part (a), a dual has a vertex of degree 4 or less. The proof that 4 colors are sufficient to properly color the vertices of such a dual is analogous to the proof of Theorem 9-3, with the above result replacing the result of Example 8-4.

□

9-11. Show that no planar graph with n vertices has a chromatic polynomial that is of the form

$$P(\lambda) = \lambda(\lambda^2 - 6\lambda + 5)Q(\lambda)$$

where $Q(\lambda)$ is a polynomial of degree n-3.

$$P(\lambda) = \lambda(\lambda-1)(\lambda-5)Q(\lambda)$$

$$P(5) = 0$$

But, by Theorem 9-3, $P(5) \geq 1$ for any planar graph.

□

9-12. Find the chromatic polynomial of the graph in Fig. 9P-1.

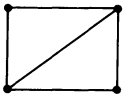
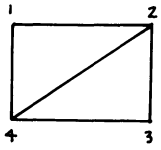


Fig. 9P-1

Define the following properties of a way of coloring the vertices of this graph with λ colors:



a_1 vertices 1 and 2 are colored the same
a_2 " 2 and 3 " " " "
a_3 " 3 and 4 " " " "
a_4 " 4 and 1 " " " "
a_5 " 2 and 4 " " " "

Then $P(\lambda) = N(a_1' a_2' \cdots a_5')$

$N = \lambda^4$

$s_1 = \binom{5}{1} \lambda^3$

$s_2 = \binom{5}{2} \lambda^2$

$s_3 = 2\lambda^2 + 8\lambda$

$s_4 = \binom{5}{4}\lambda$

$s_5 = \binom{5}{5}\lambda$

$P(\lambda) = \lambda^4 - 5\lambda^3 + 10\lambda^2 - (2\lambda^2 + 8\lambda) + 5\lambda - \lambda$

$= \lambda(\lambda^3 - 5\lambda^2 + 8\lambda - 4)$

$= \lambda(\lambda-1)(\lambda-2)^2$

□

9-13. Show that the chromatic polynomial of a tree of n vertices is $P(\lambda) = \lambda(\lambda-1)^{n-1}$.

Label the (n-1) edges with the numbers 1 through (n-1) in an arbitrary manner, and define define the following (n-1) properties of a way of coloring the vertices of the tree with λ colors:

a_1 the two vertices connected by edge 1 are colored the same
a_2 " " " " " " 2 " " " "
⋮
a_{n-1} " " " " " "(n-1)" " " "

Then $P(\lambda) = N(a_1' a_2' \cdots a_{n-1}')$

$$N = \lambda^n$$

$$s_1 = \binom{n-1}{1} \lambda^{n-1}$$

$$s_2 = \binom{n-1}{2} \lambda^{n-2}$$

$$\vdots$$

$$s_{n-1} = \binom{n-1}{n-1} \lambda$$

$$P(\lambda) = \lambda^n - \binom{n-1}{1} \lambda^{n-1} + \binom{n-1}{2} \lambda^{n-2} - \cdots + (-1)^{n-1} \binom{n-1}{n-1} \lambda$$

$$= \lambda(\lambda-1)^{n-1}$$

□

9-14. (a) Find the chromatic polynomial of a circuit of n vertices, where n is an even number.

(b) Find the chromatic polynomial of a circuit of n vertices, where n is an odd number.

Check your results by noticing that two colors are sufficient to color the vertices in a circuit of even length but are not sufficient to color the vertices in a circuit of odd length.

Define the following properties of a way of coloring the vertices of the circuit with λ colors:

a_1 vertices 1 and 2 are colored the same

a_2 " 2 and 3 " " " "

$\vdots$

a_n " n and 1 " " " "

Then $P(\lambda) = N(a_1' a_2' \cdots a_n')$

$$N = \lambda^n$$

$$s_1 = \binom{n}{1} \lambda^{n-1}$$

$$s_2 = \binom{n}{2} \lambda^{n-2}$$

$$\vdots$$

$$s_{n-1} = \binom{n}{n-1} \lambda$$

$$s_n = \binom{n}{n} \lambda$$

$$P(\lambda) = \lambda^n - \binom{n}{1}\lambda^{n-1} + \binom{n}{2}\lambda^{n-2} - \cdots + (-1)^{n-1}\binom{n}{n-1}\lambda + (-1)^n\binom{n}{n}\lambda$$

(a) n-even

$$P(\lambda) = \lambda^n - \binom{n}{1}\lambda^{n-1} + \binom{n}{2}\lambda^{n-2} - \cdots - \binom{n}{n-1}\lambda + \binom{n}{n}\lambda$$

$$= (\lambda-1)^n + (\lambda-1)$$

(b) n-odd

$$P(\lambda) = \lambda^n - \binom{n}{1}\lambda^{n-1} + \binom{n}{2}\lambda^{n-2} - \cdots + \binom{n}{n-1}\lambda - \binom{n}{n}\lambda$$

$$= (\lambda-1)^n - (\lambda-1)$$

As a check, $P(2) = \begin{cases} 2 & \text{n-even} \\ 0 & \text{n-odd} \end{cases}$

□

10-1. Use the labeling procedure to find a maximal flow in the transport network in Fig. 10P-1.

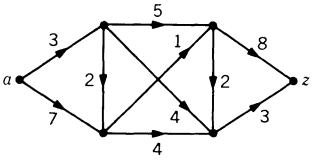
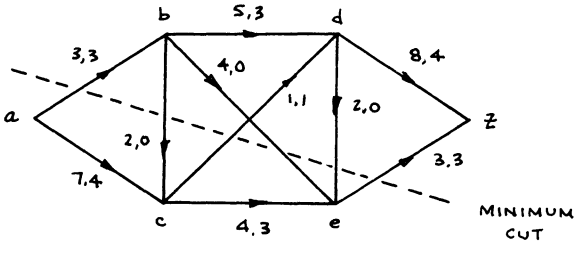


Fig. 10P-1

A maximal flow, with $\phi_V = 7$, is



□

10-2. Find a maximal flow in the transport network in Fig. 10P-2.

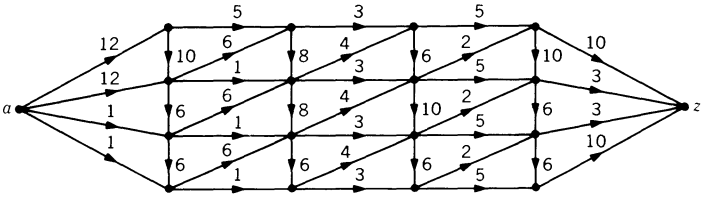
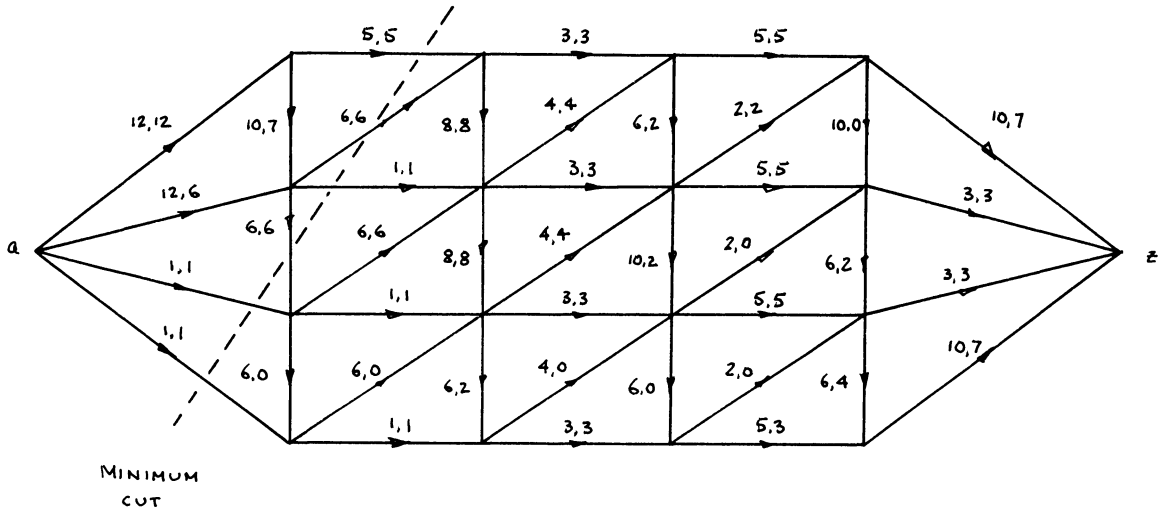


Fig. 10P-2

A maximal flow, with $\phi_V = 20$, is



10-3. Equipment is manufactured at three factories x_1, x_2, and x_3 and is to be shipped to three depots y_1, y_2, and y_3 through the transport network shown in Fig. 10P-3.

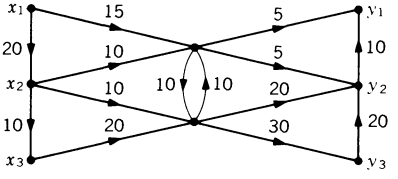
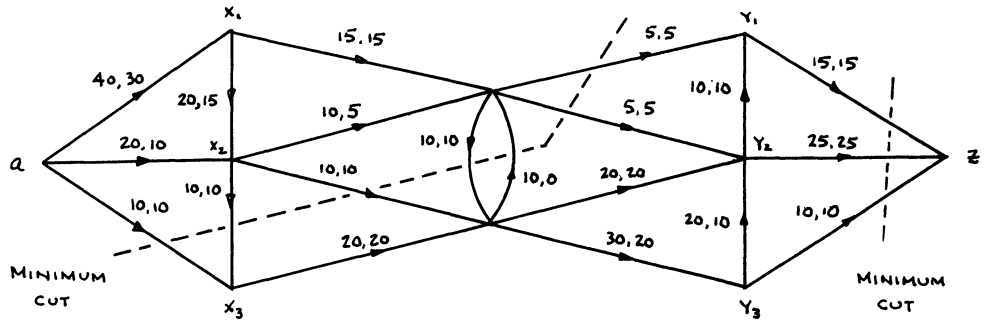


Fig. 10P-3

Factory x_1 can make 40 units, factory x_2 can make 20 units, and factory x_3 can make 10 units. Depot y_1 needs 15 units, depot y_2 needs 25 units, and depot y_3 needs 10 units. How many units should each factory make so that they can be transported to the depots?

Finding a maximal flow in the following transport network reveals that the demands of all three depots can be met by having factory x_1 make 30 units, x_2 10 units and x_3 10 units.



10-4. Find a maximal flow in the transport network in Fig. 10P-4 in which flows in the unoriented edges can be in either direction.

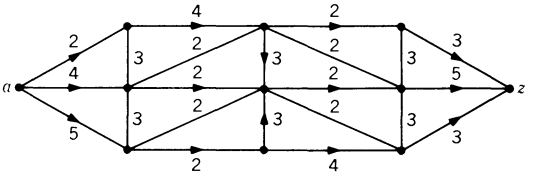
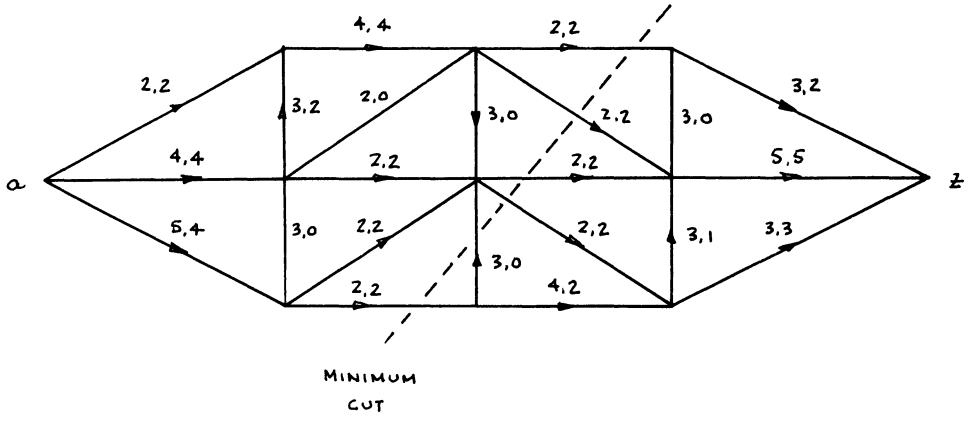


Fig. 10P-4

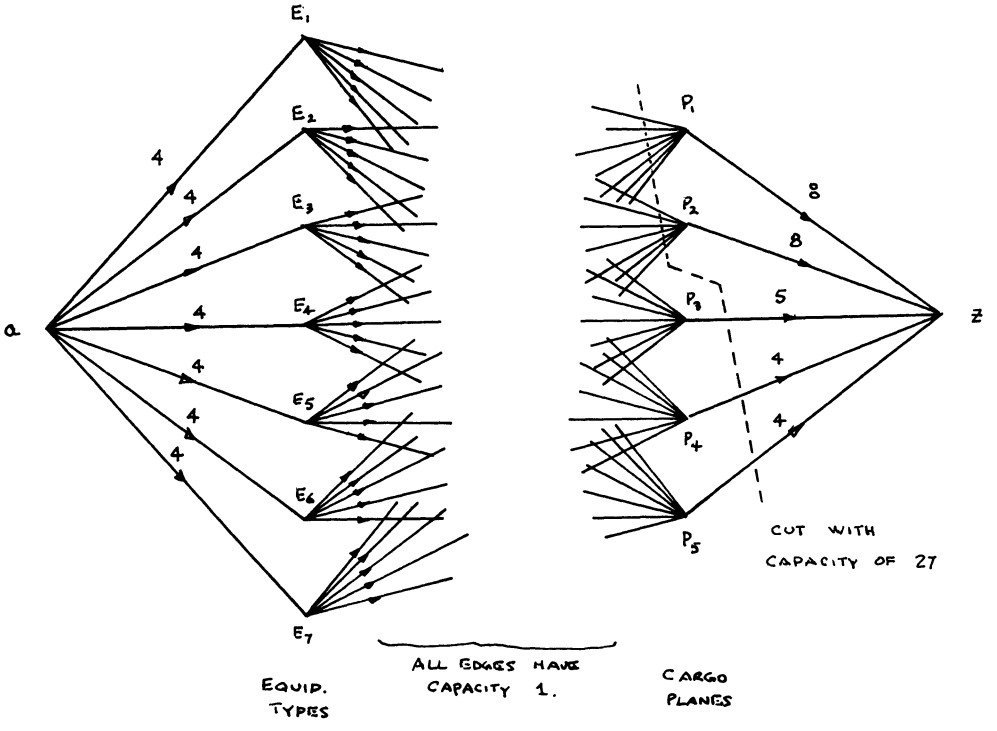
$\phi_V = 10$



10-5. (a) Seven kinds of military equipment are to be flown to a destination by five cargo planes. There are four units of each kind, and the five planes can carry eight, eight, five, four, and four units, respectively. Can the equipment be loaded in such a way that no two units of the same kind are on one plane?

(b) Give a solution to part (a) when the capacities of the planes are seven, seven, six, four, and four units, respectively.

(a)



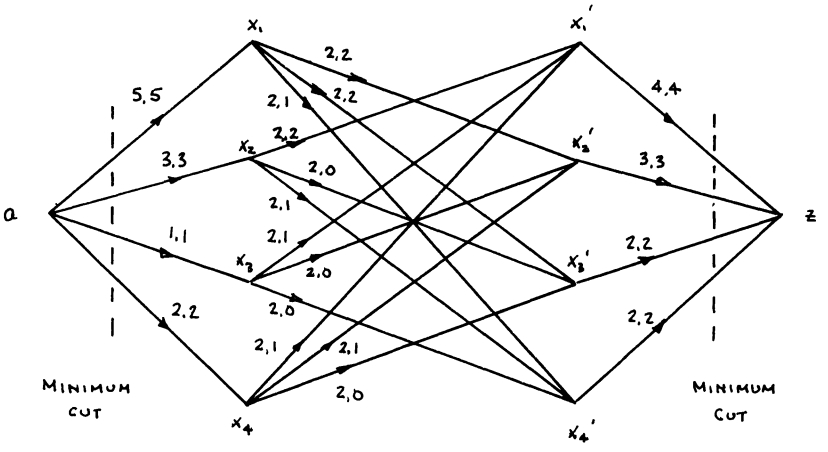
A flow in the above transport network specifies a loading plan for some or all of the equipment such that no two similar units are on the same plane. Since the capacity of the cut shown above is 27, there can be no flow having value 28. Thus, not all the equipment can be so loaded.

(b) Applying the labeling procedure to the transport network of part (a), with capacities of edges incident with the sink modified, yields the following loading plan as one possible solution.

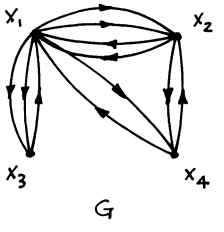
	P_1	P_2	P_3	P_4	P_5
E_1	x	x	x	x	
E_2	x	x	x	x	
E_3	x	x	x	x	
E_4	x	x	x		x
E_5	x	x	x		x
E_6	x	x	x		x
E_7	x	x		x	x

□

10-6. Construct a directed 2-graph with four vertices such that the outgoing and incoming degrees of the vertices are (5,4), (3,3), (1,2), and (2,2), respectively, by solving the corresponding network flow problem.



$\phi(x_i, x_j')$ indicates the number of directed edges in G from x_i to x_j



□

10-7. In the transport network in Fig. 10P-5, not only are there upper bounds on the flows in the edges, but there are also upper bounds on the total flows into the intermediate vertices. Find a maximal flow in the network.

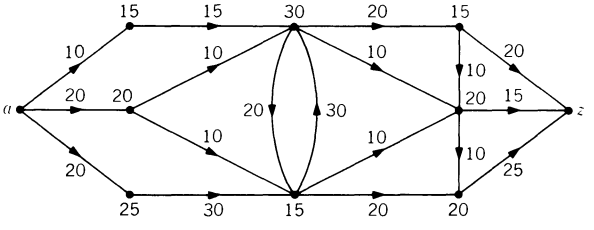
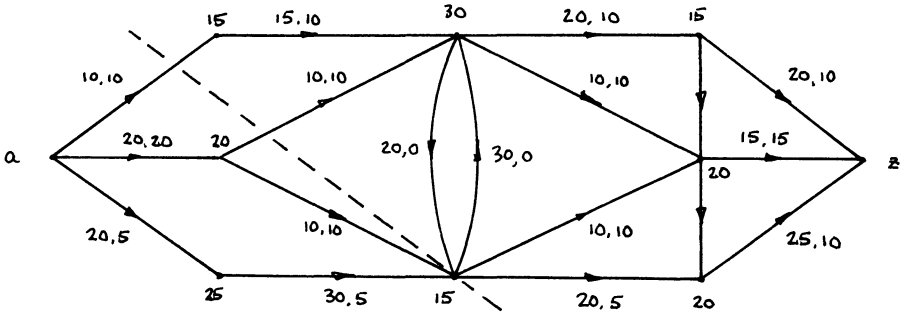


Fig. 10P-5

$\phi_V = 35.$



□

10-8. In the network in Fig. 10P-6 the numbers associated with the edges are the distances between vertices. Find a shortest path between a and z.

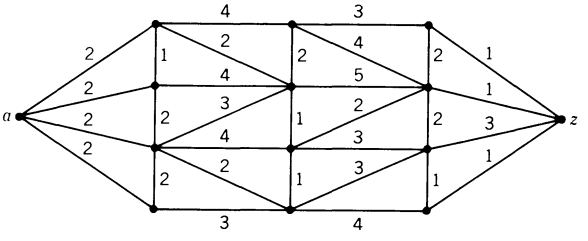
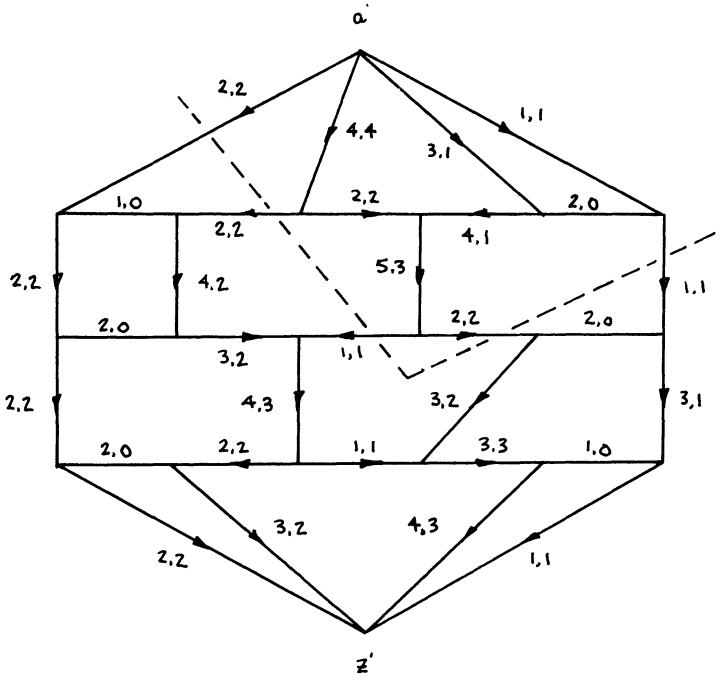


Fig. 10P-6

Find a maximal flow in the dual.



$\phi_V = 8$.

The edges in the original network corresponding to the minimum cut above form a shortest path of length 8.

□

10-9. Given a transport network, we wish to find a flow such that the flow in each edge is larger than or equal to the capacity of the edge and the value of the flow is minimal.

(a) Define the "capacity of a cut," and state the minimal-flow maximal-cut theorem which is analogous to the maximal-flow minimal-cut theorem proved in Sec. 10-3.

(b) Prove the minimal-flow maximal-cut theorem by designing an algorithm to find a minimal flow.

(a) Define the capacity of a cut as

$$\alpha(P, \bar{P}) = \begin{cases} \sum_{\substack{i \in P \\ j \in \bar{P}}} \alpha(i,j) & \text{if there are no edges from } \bar{P} \text{ to } P \\ 0 & \text{otherwise} \end{cases}$$

A minimal flow-maximal cut theorem:

In a transport network in which the flow in each edge is lower bounded only, the minimal value that a flow can achieve is equal to the maximal value of the capacities of the cuts in the network.

(b) Modified labeling procedure.

Assume an initial feasible flow.

Label the sink z with $(-,\infty)$

Label a vertex y adjacent to z with $(z^-, \delta(y))$

where $\delta(y) = \phi(y,z) - \alpha(y,z)$, if $\phi(y,z) > \alpha(y,z)$.

Do not label otherwise.

Label a vertex x adjacent to a labeled vertex y with $(y^-, \delta(x))$

where $\delta(x) = \min [(\phi(x,y) - \alpha(x,y)), \delta(y)]$, if $\phi(x,y) > \alpha(x,y)$.

Do not label otherwise.

Label a vertex w adjacent from a labeled vertex y with $(y^+, \delta(w))$

where $\delta(w) = \delta(y)$.

(It is always possible to label a vertex adjacent from a labeled vertex).

Each time the source a is labeled, reduce the flow in accordance with the labels and begin again.

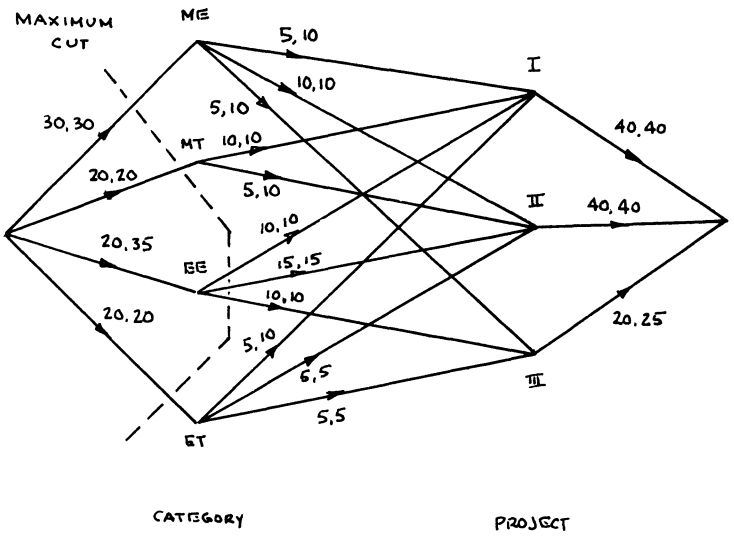
The algorithm terminates when the source cannot be labeled. At that point, if the unlabeled vertices are denoted P, and the labeled vertices $\overline{P}$, then the cut $(P, \overline{P})$ is a maximal cut with no edges from $\overline{P}$ to P and each edge from P to $\overline{P}$ carrying a minimum flow.

□

10-10. Engineers and technicians are to be hired by a company to participate in three projects. The personnel requirements of these three projects are listed in the following table:

	Minimal number of people needed in each project	Minimal number in each category			
		Mechanical engineers	Mechanical technicians	Electrical engineers	Electrical technicians
Project I	40	5	10	10	5
Project II	40	10	5	15	5
Project III	20	5	0	10	5

Moreover, to prepare for later expansion, the company wants to hire at least 30 mechanical engineers, 20 mechanical technicians, 20 electrical engineers, and 20 electrical technicians. What is a minimal number of persons in each category that the company should hire, and how should they be allocated to the three projects?



10-11. In the graph in Fig. 10P-7 a minimal set of edges is to be selected such that every vertex is incident with at least one of the edges in the set. Solve this problem as a minimal flow problem associated with a transport network.

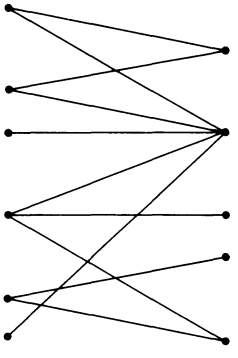
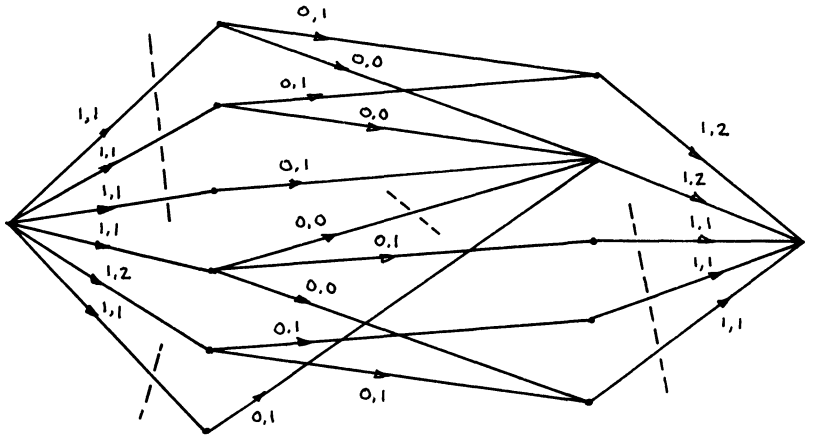


Fig. 10P-7



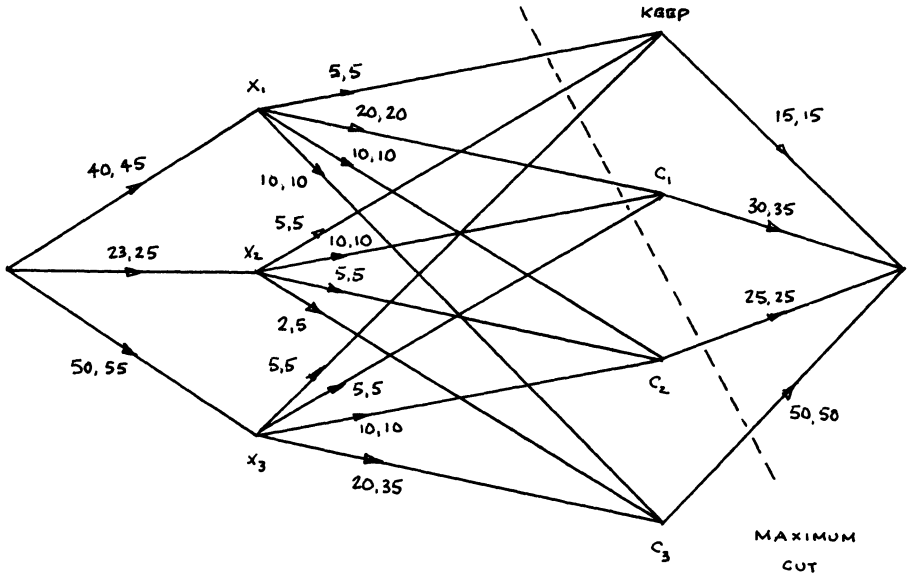
Maximum cut is indicated by dashed lines.

□

10-12. Three candidates x_1, x_2, and x_3 have been promised minimal amounts of campaign money of \$40,000, \$23,000, and \$50,000, respectively. Each candidate in turn has promised three campaign areas at least the amounts of money shown in the table below. In addition, each candidate will need at least \$5,000 for his own expenses.

Candidate	Campaign area		
	C_1	C_2	C_3
x_1	\$20,000	\$10,000	\$10,000
x_2	10,000	5,000	2,000
x_3	5,000	10,000	20,000

If the three campaign areas C_1, C_2, and C_3 require a minimum of \$30,000, \$25,000, and \$50,000, respectively, to conduct a thorough campaign, what is a minimal amount of campaign money each candidate must obtain and how should it be distributed?



All capacities and flows are in thousands of dollars.

□

10-13. Show that no feasible flow exists in the transport network G in Fig. 10P-8 by

(a) using an exhaustive argument

(b) finding two cuts $(P,\overline{P})$ and $(Q,\overline{Q})$ such that

$$\beta(Q,\overline{Q}) - \alpha(\overline{Q},Q) > \alpha(P,\overline{P}) - \beta(\overline{P},P)$$

(c) constructing $\hat{G}$ and finding a maximal flow in $\hat{G}$

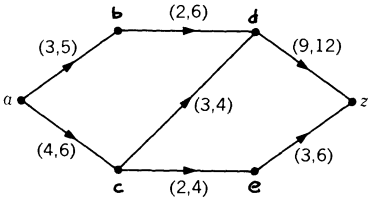


Fig. 10P-8

(a) If there is a feasible flow in G, then $\phi(c,d) = 3$ or 4.

Assume $\phi(c,d) = 3$.

Then $\phi(b,d) = 6$ and $\phi(a,b) = 6$ which is impossible.

Assume $\phi(c,d) = 4$.

Then $\phi(a,c) = 6$, $\phi(c,e) = 2$, and $\phi(e,z) = 2$ which is impossible.

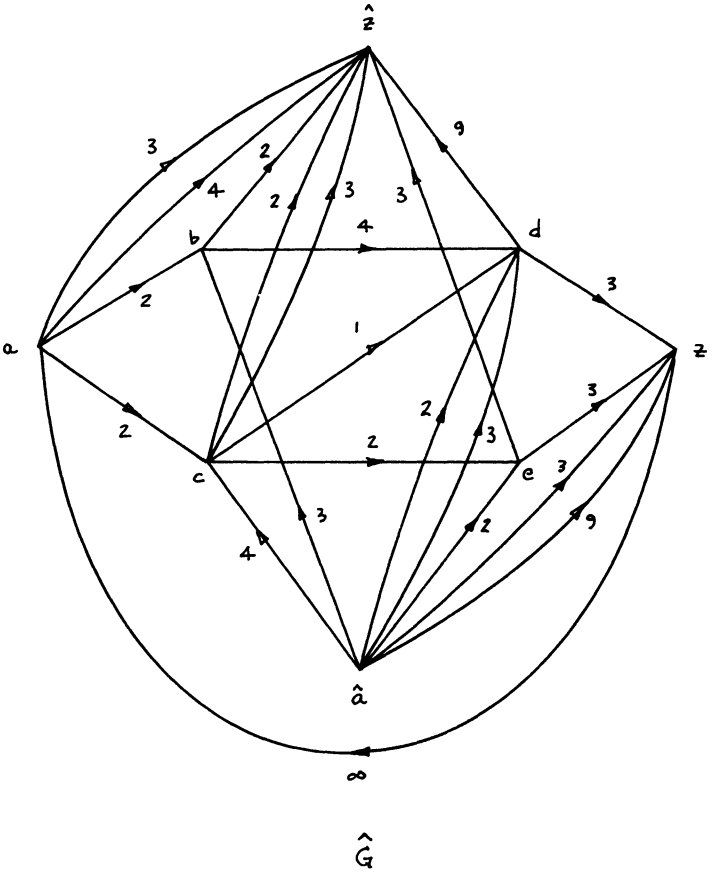
(b) Let $(P,\overline{P})$ = (a)(bcdez)

$(Q,\overline{Q})$ = (abcde)(z)

$\beta(Q,\overline{Q}) - \alpha(\overline{Q},Q) = 12$

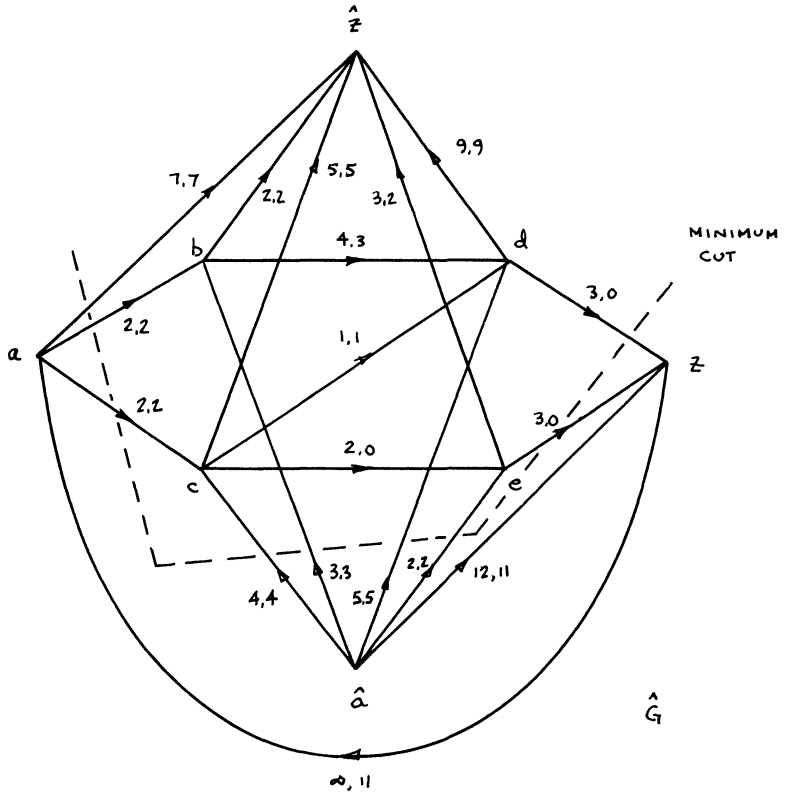
$\alpha(P,\overline{P}) - \beta(\overline{P},P) = 11$

(c)



$\hat{G}$

Combine parallel edges in $\hat{G}$ and find a maximal flow.



Since a maximal flow in $\hat{G}$ does not saturate all edges incident with $\hat{z}$, there is no feasible flow in G.

□

10-14. Find a maximal flow in the transport network in Fig. 10P-9.

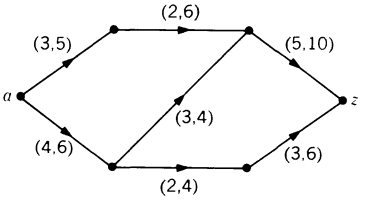
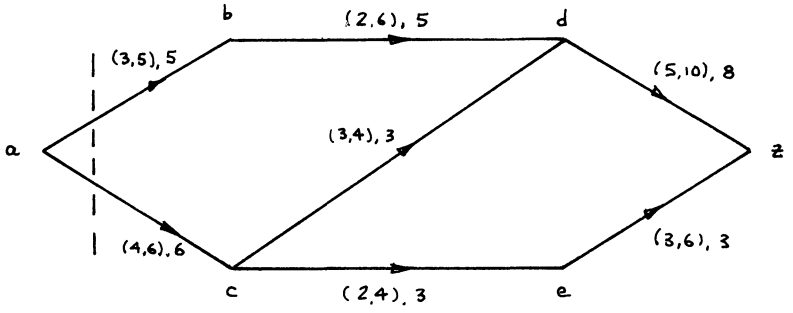


Fig. 10P-9



cut for which
$\alpha(P,\overline{P}) - \beta(\overline{P},P)$
is minimum.

□

10-15. (a) State a procedure for finding a minimal flow in a transport network where there is a lower bound as well as an upper bound on the flow in every edge.

(b) Find a minimal flow in the transport network in Fig. 10P-9.

(a) In a transport network where there are both upper and lower bounds on the values of the flows in the edges, the value of the minimal feasible flow is equal to the maximal value of $\beta(P,\overline{P}) - \alpha(\overline{P},P)$ for all cuts $(P,\overline{P})$.

Assume an initial feasible flow.

Label the sink z with $(-,\infty)$

Label a vertex y adjacent to z with $(z^-, \delta(y))$

where $\delta(y) = \phi(y,z) - \beta(y,z)$, if $\phi(y,z) > \beta(y,z)$.

Do not label otherwise.

Label a vertex x adjacent to a labeled vertex y with $(y^-, \delta(x))$

where $\delta(x) = \min[(\phi(x,y) - \beta(x,y)), \delta(y)]$, if $\phi(x,y) > \beta(x,y)$.

Do not label otherwise.

Label a vertex w adjacent from a labeled vertex y with $(y^+, \delta(w))$

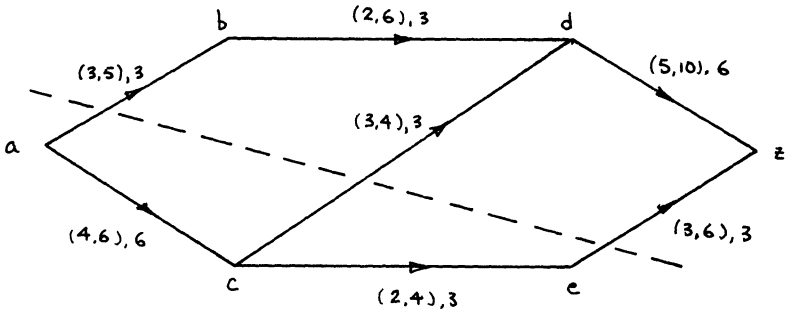
where $\delta(w) = \min[(\alpha(y,w) - \phi(y,w)), \delta(y)]$, if $\phi(y,w) < \alpha(y,w)$.

Do not label otherwise.

Each time the source a is labeled, reduce the flow in accordance with the labels.

At the termination of this labeling procedure, the value of the flow is equal to $\beta(P,\overline{P}) - \alpha(\overline{P},P)$ where P is the set of unlabeled vertices and $\overline{P}$ is the set of labeled vertices.

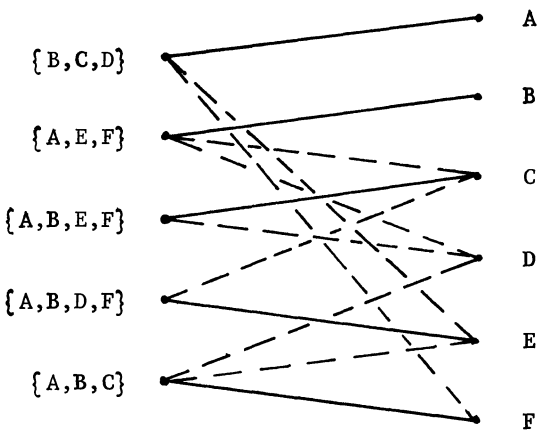
(b)



cut for which
$\beta(P,\overline{P}) - \alpha(\overline{P},P)$
is maximum.

□

11-1. Six senators A, B, C, D, E, and F are members of five committees. The memberships of the committees are {B,C,D}, {A,E,F}, {A,B,E,F}, {A,B,D,F}, and {A,B,C}. The activities of each committee are to be reviewed by a senator who is not on the committee. Can five distinct senators be selected? How?



□

11-2. There are rs couples at a dance. The men are divided into r groups, with s men in each group, according to their ages. The women are also divided into r groups, with s women in each group, according to their heights. Show that r couples can be selected such that every age group and every height group will be represented.

Form a bipartite graph G in which X contains r vertices $a_1, a_2, \cdots, a_r$, one for each male age group, and Y contains r vertices $h_1, h_2, \cdots, h_r$, one for each female height group. There is an edge (a_i, h_j) iff there is a couple composed of a man in age group i and a woman in height group j.

The desired selection of couples can be made iff there is a complete matching in G.

Corresponding to any $A \subseteq X$, there are $|A| \cdot s$ men coupled with a subset of $|R(A)| \cdot s$ women, so that

$$|A| \cdot s \leq |R(A)| \cdot s$$

or

$$|A| \leq |R(A)|$$

and by Theorem 11-1 there is a complete matching in G.

□

11-3. (a) Let π_1 and π_2 be two partitions on a set of m elements, both of which contain exactly r disjoint subsets. State a necessary and sufficient condition for the possibility of selecting r of the m elements such that the r disjoint subsets in π_1 as well as the r disjoint subsets in π_2 are represented.

(b) The numbers 1, 2, ···, 20 are partitioned into four disjoint subsets according to their remainders when they are divided by 4. They are also partitioned into four disjoint subsets according to the number of prime factors they contain, namely, {{1,2,3,5,7,11,13,17,19}, {4,6,9,10,14,15}, {8,12,18,20}, {16}}. Is it possible to select four numbers such that there is a representative for each possible remainder when divided by 4 and a representative for each possible number of prime factors?

(a) Let $\pi_1 = (X_1)(X_2) \cdots (X_r)$

$\pi_2 = (Y_1)(Y_2) \cdots (Y_r)$

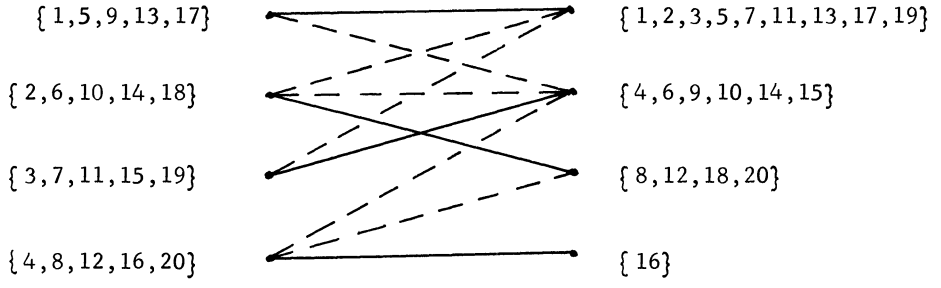
Form a bipartite graph G in which the vertices in X correspond to the disjoint subsets $X_1, X_2, \cdots X_r$ and the vertices in Y correspond to the disjoint subsets $Y_1, Y_2, \cdots Y_r$. Let there be an edge (X_i, Y_j) iff $X_i \cap Y_j \neq \phi$. The desired selection of elements can be made iff there is a complete matching in G.

For each union of k of the subsets $X_1, X_2, \cdots X_r$, let n be the least number of subsets $Y_1, Y_2, \cdots Y_r$ whose union contains the former.

Applying Theorem 11-1, with $|A| = k$ and $|R(A)| = n$, the condition is:

$k \leq n$ for all unions of k of the subsets $X_1, X_2, \cdots X_r$, $k = 0, 1, \cdots, r$.

(b) Using the construction of part (a):



One possible selection is {15,16,17,18}.

□

11-4. A factor of an undirected graph (containing no loops) is defined as a circuit or an edge-disjoint union of circuits that contains all vertices of the graph. (If a factor is a circuit, it is a Hamiltonian circuit.) Show that a necessary and sufficient condition for a graph to possess a factor is that every subset of vertices A has the property $|A| \leq |R(A)|$ where R(A) is the set of vertices that are adjacent to one or more vertices in A.

Given a graph G with vertex set V, construct a bipartite graph G' in which the disjoint subsets of vertices are denoted V_x and V_y, where

$$V_x = V_y = V$$

and, for all $v_i \in V_x$, $v_j \in V_y$

$$(v_i, v_j) \in G' \quad \text{iff} \quad (v_i, v_j) \in G$$

Factors of G correspond to complete matchings in G'. (A complete matching in G' can be viewed as a permutation on the set V, the cycles of which correspond to circuits in G. The condition that G contain no loops guarantees that every cycle is of length 2 or more.)

The result follows by application of Theorem 11-1.

□

11-5. A group of p boys $x_1, x_2, \cdots, x_p$ are ranked according to their unpopularities; that is, x_1 knows the same number or fewer girls than x_2 does, x_2 knows the same number or fewer girls than x_3 does, and so on. A group of q girls $y_1, y_2, \cdots, y_q$ are ranked according to their popularities; that is, y_1 knows the same number or more boys than y_2 does, y_2 knows the same number or more boys than y_3 does, and so on. Show that the p boys can be matched with p of the q girls if the number of girls that the least popular k boys know exceeds the number of boys that the most popular k-1 girls know for $1 \leq k \leq p$.

Show there is a complete matching in a bipartite graph G in which

$$X = \{x_1, x_2, \cdots, x_p\}$$

$$Y = \{y_1, y_2, \cdots, y_q\}$$

and

$$(x_i, y_j) \in G \quad \text{iff} \quad x_i \text{ knows } y_j.$$

For any $A \subseteq X$ containing k boys, k = 1, 2, ···, p, let

$$A = \{x_{i_1}, x_{i_2}, \cdots, x_{i_k}\}$$

Assume R(A) contains (k-1) or fewer girls

$$R(A) \subseteq \{y_{j_1}, y_{j_2}, \cdots, y_{j_{k-1}}\}.$$

Let $q(x_i)$ = number of girls x_i knows

$b(y_j)$ = number of boys y_j knows

Then, by counting edges in the subgraph of G containing vertices A, R(A) and all edges connecting pairs of these,

$$g(x_{i_1}) + g(x_{i_2}) + \cdots + g(x_{i_k}) \leq b(y_{j_1}) + b(y_{j_2}) + \cdots + b(y_{j_{k-1}}) \qquad [1]$$

Since $b(y_1) \geq b(y_2) \geq \cdots \geq b(y_q)$

$$b(y_{j_1}) + b(y_{j_2}) + \cdots + b(y_{j_{k-1}}) \leq b(y_1) + b(y_2) + \cdots + b(y_{k-1}) \qquad [2]$$

Also, since $g(x_1) \geq g(x_2) \leq \cdots \leq g(x_p)$

$$g(x_{i_1}) + g(x_{i_2}) + \cdots + g(x_{i_k}) \geq g(x_1) + g(x_2) + \cdots + g(x_k) \qquad [3]$$

Combining [1], [2], and [3]:

$$g(x_1) + g(x_2) + \cdots + g(x_k) \leq b(y_1) + b(y_2) + \cdots + b(y_{k-1})$$

which contradicts the given condition.

Therefore, R(A) contains k or more girls, and by Theorem 11-1, there is a complete matching in G.

□

11-6. Let G be a bipartite graph with X and Y being the two disjoint sets of vertices. Show that $\beta(G) = |Y| + \delta(G)$ where $\beta(G)$ is the independence number and $\delta(G)$ is the deficiency of the graph G.

Any $A \subseteq X$ is an independent set.

No vertex in Y - R(A) is adjacent to a vertex in A, nor are any two vertices in Y - R(A) adjacent. Thus, $A \cup Y - R(A)$ is an independent set. Furthermore, any maximal independent set in G can be expressed in this form.

$$\begin{aligned}\beta(G) &= \max |A \cup Y - R(A)| \\ &= \max (|A| + |Y| - |R(A)|) \\ &= |Y| + \max (|A| - |R(A)|) \\ &= |Y| + \delta(G)\end{aligned}$$

□

11-7. In a bipartite graph G with X and Y being the two disjoint set of vertices, let A be a subset of X such that $\delta(A) = \delta(G)$. Show that $\delta(X - A) \leq 0$.

Apply Lemma 11-1

$$\delta(A_1 \cup A_2) + \delta(A_1 \cap A_2) \geq \delta(A_1) + \delta(A_2)$$

with $A_1 = A$

$A_2 = X - A$

to obtain

$$\delta(x) + 0 \geq \delta(A) + \delta(X - A)$$

$$\delta(X - A) \leq \delta(X) - \delta(A) = \delta(X) - \delta(G)$$

and since

$$\delta(X) \leq \delta(G)$$

$$\delta(X - A) \leq 0$$

□

11-8. Let G be a bipartite graph with X and Y being the two disjoint sets of vertices. If there are four or more edges incident from each vertex in X and there are five or less edges incident into each vertex in Y, show that $\delta(G) \leq 2$ for $|X| \leq 10$.

For any $A \subseteq X$ consider the subgraph of G consisting of the vertices in A, those in R(A) and edges between the two.

Let e be the number of edges in this subgraph. Then,

$$e \geq 4|A|$$

$$e \leq 5|R(A)|$$

$$5|R(A)| \geq 4|A|$$

$$\delta(A) = |A| - |R(A)|$$

$$\leq |A| - \frac{4}{5}|A| = \frac{1}{5}|A| \leq 2$$

and since

$$\delta(G) = \max \delta(A)$$

$$\delta(G) \leq 2$$

□

11-9. Let G be a directed graph with n vertices for which the incoming and outgoing degree of each vertex is k. Show that G can be decomposed into edge-disjoint circuits.
HINT: In the $n \times n$ (0,1)-matrix $A = [a_{ij}]$ defined as

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in G \\ 0 & \text{otherwise} \end{cases}$$

each row and column contains k 1's. Show that A can be decomposed as a sum of permutation matrices.

There are n 1's in A no two of which are in the same line. Otherwise, by the König - Egerváry Theorem (Example 11-6), there would be ℓ lines, $\ell < n$, containing all 1's. But then, summing all elements of A,

$$nk \leq \ell k < nk$$

which is a contradiction.

Let P_1 be an $n \times n$ permutation matrix whose 1's coincide with these n 1's of A. $A - P_1$ is an $n \times n$ (0,1)-matrix in which every line contains (k-1) 1's.

Repeating this process (k-1) more times gives

$$A = P_1 + P_2 + \cdots + P_k$$

where $P_1, P_2, \cdots, P_k$ are permutation matrices, no two of which contain a 1 in the same position.

Each permutation matrix $P = [p_{ij}]$ specifies a set of n directed edges of G, namely

$$\{(v_i, v_j) \in G \mid p_{ij} = 1\}$$

These edges form a circuit or edge-disjoint union of circuits in G, one for each cycle of the permutation π, on the set of vertices of G, defined by:

$$\pi(v_i) = v_j \quad \text{iff} \quad p_{ij} = 1.$$

□

11-10. Let A be an $m \times n$ (0,1)-matrix ($m \leq n$). If each row in A has exactly k 1's and each column in A has k or less 1's, show that

$$A = P_1 + P_2 + \cdots + P_k$$

where the P's are $m \times n$ (0,1)-matrices that have a 1 in each row and no more than one 1 in each column.

The sum of each row of A is k.

If m = n, the sum of each column of A is k. Let A' = A.

Otherwise, using elements from $\{0, 1, 2, \cdots, k\}$ add n-m rows to A to obtain an $n \times n$ matrix A' in which the sum of every line (row or column) is k.

There are n non-zero entries in A' no two of which are in the same line. Otherwise, by the König-Egerváry theorem (Example 11-6), there would be ℓ lines, $\ell < n$, containing all non-zero entries. But then, summing all entries

$$nk \leq \ell k < nk$$

which is a contradiction.

Let P_1' be an $n \times n$ permutation matrix whose 1's coincide with these n non-zero entries of A'. Then $A' - P_1'$ is an $n \times n$ matrix in which the sum of every line is (k-1).

Repeating this process (k-1) more times gives

$$A' = P_1' + P_2' + \cdots + P_k'$$

Discarding the added rows (if any) from A' and the corresponding rows from $P_1', P_2', \cdots, P_k'$ gives the desired decomposition of A.

□

11-11. In this problem, a method of constructing a maximal matching in a bipartite graph, known as the Hungarian method, is studied. Let G be a bipartite graph, and let W denote the set of edges in a matching from X to Y. A vertex (in X or Y) is said to be unsaturated if no edge incident with it is included in W. A path in G is said to be an alternate path if it contains edges in W and E - W alternately, where E denotes the set of edges in G.

The Hungarian method is based on the following result: A matching is maximal if and only if there is no alternate path connecting two distinct unsaturated vertices.

(a) Show that a matching is not maximal if there exists such a path by showing that for each such path, the number of vertices in X matched into Y can be increased by 1.

(b) Label some of the vertices of G as follows:

1. Every unsaturated vertex in X is labeled.

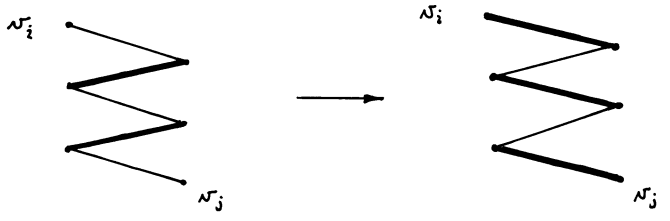
2. Any vertex (in X or Y) connected to an unsaturated vertex in X by an alternate path is labeled.

What is the deficiency of the subset of X that contains all the labeled vertices in X? Show that a matching is maximal if no alternate path connecting two distinct unsaturated vertices exists.

(c) Suggest a construction procedure for finding a maximal matching in a bipartite graph, based on the above result.

Consider the edges in W to be heavy edges, and those in E-W to be light edges.

(a) Suppose there exists an alternate path between unsaturated vertices v_i and v_j. Since the first and last edge in the path are light, the path must be of odd length and v_i and v_j are not both in X or Y. The number of light edges in the path is one larger than the number of heavy edges. Deleting these heavy edges from W and adding these light edges to W leaves every vertex in the path saturated, and increases the number of vertices in X matched into Y by one. For example:



(b) Let $A \subseteq X$ consist of all labeled vertices in X. The set of labeled vertices in Y is equal to R(A), which may be seen as follows.

Every labeled vertex in Y is in R(A), otherwise it would not be labeled.

Every vertex in R(A) is a labeled vertex in Y. For any vertex v_y in R(A), adjacent to some labeled vertex v_x in A, the edge (v_x, v_y) if light would cause v_y to be labeled, or if heavy would require v_y to be labeled as a prerequisite for the labeling of v_x.

Every vertex in R(A) (i.e., the labeled vertices in Y) is saturated, otherwise there would be an alternate path connecting two distinct unsaturated vertices.

Furthermore, the set of vertices in A adjacent via heavy edges to the set R(A), is just the set of saturated vertices in A.

Thus,

$$\delta(A) = |A| - |R(A)| = \text{number of unsaturated vertices in X.}$$

The set W of heavy edges matches $|X| - \delta(A)$ vertices in X into Y.
Since

$$\delta(A) \leq \delta(G)$$

it follows that

$$|X| - \delta(A) \geq |X| - \delta(G)$$

Then, by Theorem 11-2

$$|X| - \delta(A) = |X| - \delta(G)$$

and W is a maximal matching in G.

(c) (c) Look for alternate paths connecting two distinct unsaturated vertices. Each time such a path is found, the matching can be increased by one as in part (a). If no such path exists, the matching is maximal, as shown in part (b).

12-1. Maximize the function $C = 3x_1 + 6x_2 + 2x_3$ subject to the constraints

$$3x_1 + 4x_2 + x_3 \leq 2$$

$$x_1 + 3x_2 + 2x_3 \leq 1$$

$$x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0$$

Adding slack variables

$$3x_1 + 4x_2 + x_3 + x_4 = 2$$

$$x_1 + 3x_2 + 2x_3 + x_5 = 1$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

A basic feasible solution is (0,0,0,2,1).

Applying the simplex method

	x_1	x_2	x_3	
x_4	3	4	1	2
x_5	1	**3**	2	1
C	-3	-6	-2	0

	x_1	x_5	x_3	
x_4	$\frac{5}{3}$	$-\frac{4}{3}$	$-\frac{5}{3}$	$\frac{2}{3}$
x_2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$
C	-1	2	2	2

	x_4	x_5	x_3	
x_1	$\frac{3}{5}$	$-\frac{4}{5}$	-1	$\frac{2}{5}$
x_2	$-\frac{1}{5}$	$\frac{3}{5}$	1	$\frac{1}{5}$
C	$\frac{3}{5}$	$\frac{6}{5}$	1	$\frac{12}{5}$

The optimal solution is $(\frac{2}{5}, \frac{1}{5}, 0, 0, 0)$ for which

$$C = \frac{12}{5}$$

□

12-2. Use the simplex method to show that the value of the objective function in the following linear programming problem is unbounded. Maximize the function $C = 3x_1 + 4x_2$ subject to the constraints

$$x_1 + 2x_2 \geq 6$$

$$2x_1 + x_2 \geq 6$$

$$x_1 \geq 0 \qquad x_2 \geq 0$$

(Note: $x_1 = 2$ and $x_2 = 2$ is a basic feasible solution.)

Adding slack variables

$$x_1 + 2x_2 - x_3 = 6$$

$$2x_1 + x_2 - x_4 = 6$$

$$x_1, x_2, x_3, x_4 \geq 0$$

A basic feasible solution is (2,2,0,0).

Re-expressing the constraints and the objective function in terms of the basic variables x_3 and x_4,

$$x_1 + \frac{1}{3}x_3 - \frac{2}{3}x_4 = 2$$

$$x_2 - \frac{2}{3}x_3 + \frac{1}{3}x_4 = 2$$

$$C - \frac{5}{3}x_3 - \frac{2}{3}x_4 = 14$$

	x_3	x_4	
x_1	$\boxed{\frac{1}{3}}$	$-\frac{2}{3}$	2
x_2	$-\frac{2}{3}$	$\frac{1}{3}$	2
C	$-\frac{5}{3}$	$-\frac{2}{3}$	14

	x_1	x_4	
x_3	3	-2	6
x_2	2	-1	6
C	5	-4	24

There is no bound on the increment to x_4.

C is unbounded.

□

12-3. Minimize the function $C = x_1 - x_2$ subject to the constraints

$$4x_1 + 5x_2 \le 10$$

$$5x_1 + 2x_2 \le 1$$

$$3x_1 + 8x_2 \le 12$$

$$x_1 \ge 0 \qquad x_2 \ge 0$$

Use the simplex method and start with the basic feasible solution $x_1 = 0$ and $x_2 = 0$.

$$4x_1 + 5x_2 + x_3 = 10$$

$$5x_1 + 2x_2 + x_4 = 1$$

$$3x_1 + 8x_2 + x_5 = 12$$

$$x_1, x_2, x_3, x_4, x_5 \ge 0$$

Maximize

$$C' = -x_1 + x_2$$

A basic feasible solution is (0, 0, 10, 1, 12)

	x_1	x_2	
x_3	4	5	10
x_4	5	[2]	1
x_5	3	8	12
C'	1	-1	0

	x_1	x_4	
x_3	$-\frac{17}{2}$	$-\frac{5}{2}$	$\frac{15}{2}$
x_2	$\frac{5}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
x_5	-17	-4	8
C'	$\frac{7}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

The optimal solution is $(0, \frac{1}{2}, \frac{15}{2}, 0, 8)$ for which

$$C' = \frac{1}{2}$$

$$C = -\frac{1}{2}$$

□

12-4. Maximize the function $C = 3x_1 + 4x_2 - 2x_3$ subject to the constraints

$$3x_1 + 4x_2 + x_3 \le 2$$

$$2x_1 - 3x_2 + x_3 \ge -4$$

$$x_1 \le 0 \qquad x_2 \ge 0 \qquad x_3 \le 0$$

Let $y_1 = -x_1$, $y_2 = x_2$, $y_3 = -x_3$. The problem then is to maximize

$$C = -3y_1 + 4y_2 + 2y_3$$

subject to

$$-3y_1 + 4y_2 - y_3 \leq 2$$

$$-2y_1 - 3y_2 - y_3 \geq -4$$

$$y_1, y_2, y_3 \geq 0$$

Adding slack variables

$$-3y_1 + 4y_2 - y_3 + y_4 = 2$$

$$-2y_1 - 3y_2 - y_3 - y_5 = -4$$

$$y_1, y_2, y_3, y_4, y_5 \geq 0$$

A basic feasible solution is (0,0,0,2,4).
Applying the simplex method

	y_1	y_2	y_3	
y_4	-3	4	-1	2
y_5	2	3	[1]	4
C	3	-4	-2	0

	y_1	y_2	y_5	
y_4	-1	7	1	6
y_3	2	3	1	4
C	7	2	2	8

The optimal solution is (0,0,4,6,0) for which C = 8. In terms of x_1, x_2, x_3, the optimal solution is $x_1 = x_2 = 0$, $x_3 = -4$.

□

12-5. Maximize the function $C = 5x_1 + 2x_2$ subject to the constraints

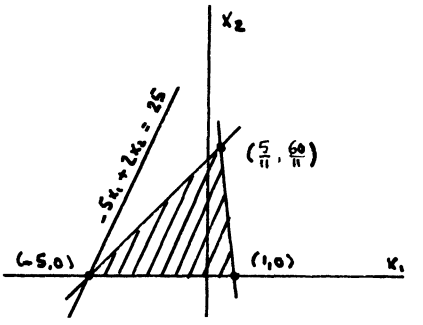
$$-x_1 + x_2 \leq 5$$
$$10x_1 + x_2 \leq 10$$
$$x_2 \geq 0$$

by

(a) Using a geometric method

(b) Formulating the problem as a linear programming problem and using the simplex method to solve it.

(a)



The optimal solution is (-5,0), for which C = 25.

(b) Let $x_1 = x_1' - x_1''$, where $x_1' \geq 0$ and $x_1'' \geq 0$. The problem then is to maximize

$$C = -5x_1' + 5x_1'' + 2x_2$$

subject to

$$-x_1' + x_1'' + x_2 \leq 5$$
$$10x_1' - 10x_1'' + x_2 \leq 10$$
$$x_1', x_1'', x_2 \geq 0$$

Adding slack variables

$$-x_1' + x_1'' + x_2 + x_3 = 5$$
$$10x_1' - 10x_1'' + x_2 + x_4 = 10$$
$$x_1', x_1'', x_2, x_3, x_4 \geq 0$$

A basic feasible solution is (0,0,0,5,10). Applying the simplex method

	x_1'	x_1''	x_2	
x_3	-1	[1]	1	5
x_4	10	-10	1	10
C	5	-5	-2	0

	x_1'	x_3	x_2	
x_1''	-1	1	1	5
x_4	0	10	11	60
C	0	5	3	25

The optimal solution is (0,5,0,0,60), for which C = 25.

□

12-6. Solve the dual of the following linear programming problem: Maximize the function $C = -3x_1 + 4x_2 - x_3$ subject to the constraints

$$-x_1 + x_2 + x_3 \leq 5$$
$$x_1 - x_2 - 2x_3 \geq -6$$
$$5x_1 + x_2 - 3x_3 = 4$$
$$x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0$$

The primal problem may be written as follows.

Maximize

$$C = -3x_1 + 4x_2 - x_3$$

subject to

$$\begin{aligned} -x_1 + x_2 + x_3 &\leq 5 \\ -x_1 + x_2 + 2x_3 &\leq 6 \\ -5x_1 - x_2 + 3x_3 &\leq -4 \\ 5x_1 + x_2 - 3x_3 &\leq 4 \end{aligned}$$

$$x_1, x_2, x_3 \geq 0$$

The dual problem is:

Minimize

$$C' = 5y_1 + 6y_2 - 4y_3 + 4y_4$$

subject to

$$\begin{aligned} -y_1 - y_2 - 5y_3 + 5y_4 &\geq -3 \\ y_1 + y_2 - y_3 + y_4 &\geq 4 \\ y_1 + 2y_2 + 3y_3 - 3y_4 &\geq -1 \end{aligned}$$

$$y_1, y_2, y_3, y_4 \geq 0$$

Solution of the primal problem.

Adding slack variables,

$$\begin{aligned} -x_1 + x_2 + x_3 + x_4 &= 5 \\ -x_1 + x_2 + 2x_3 + x_5 &= 6 \\ -5x_1 - x_2 + 3x_3 + x_6 &= -4 \\ 5x_1 + x_2 - 3x_3 + x_7 &= 4 \end{aligned}$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$$

A basic feasible solution is (0,4,0,1,2,0,0). Applying the simplex method,

	x_1	x_3	x_7	
x_2	5	-3	1	4
x_4	-6	[4]	-1	1
x_5	-6	5	-1	2
x_6	0	0	1	0
C	23	-11	4	16

	x_1	x_4	x_7	
x_2	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{19}{4}$
x_3	$-\frac{3}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$
x_5	$\frac{3}{2}$	$-\frac{5}{4}$	$\frac{1}{4}$	$\frac{3}{4}$
x_6	0	0	1	0
C	$\frac{13}{2}$	$\frac{11}{4}$	$\frac{5}{4}$	$\frac{75}{4}$

The optimal solution is $(0, \frac{19}{4}, \frac{1}{4}, 0, \frac{3}{4}, 0, 0)$, for which

$$C = \frac{75}{4}$$

Solution of the dual problem.

Adding slack variables,

$$\begin{aligned} -y_1 - y_2 - 5y_3 + 5y_4 - y_5 &= -3 \\ y_1 + y_2 - y_3 + y_4 - y_6 &= 4 \\ y_1 + 2y_2 + 3y_3 - 3y_4 - y_7 &= -1 \end{aligned}$$

$$y_1, y_2, y_3, y_4, y_4, y_5, y_6, y_7 \geq 0$$

Since there is no obvious basic feasible solution, the method of Section 12-6 (Complication 3) is applied. Introducing the new variable y_0 to the second constraint equation,

$$\begin{aligned} -y_1 - y_2 - 5y_3 + 5y_4 - y_5 &= -3 \\ y_1 + y_2 - y_3 + y_4 - y_6 + y_0 &= 4 \\ y_1 + 2y_2 + 3y_3 + 3y_4 - y_7 &= -1 \end{aligned}$$

$$y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7 \geq 0$$

Letting M be a large positive constant, the objective function may be written as

$$\text{Minimize} \quad C' = 5y_1 + 6y_2 - 4y_3 + 4y_4 + My_0$$

or

$$\text{Maximize} \quad -C' = -5y_1 - 6y_2 + 4y_3 - 4y_4 - My_0$$

A basic feasible solution is $(y_0,y_1,y_2,y_3,y_4,y_5,y_6,y_7) = (4,0,0,0,0,3,0,1)$. Applying the simplex method,

	y_1	y_2	y_3	y_4	y_6	
y_5	[1]	1	5	-5	0	3
y_0	1	1	-1	1	-1	4
y_7	-1	-2	-3	3	0	1
-C'	-M+5	-M+6	M-4	-M+4	M	-4M

	y_5	y_2	y_3	y_4	y_6	
y_1	1	1	5	-5	0	3
y_0	-1	0	-6	[6]	-1	1
y_7	1	-1	2	-2	0	4
-C'	M-5	1	6M-29	-6M+29	M	-M-15

	y_5	y_2	y_3	y_0	y_6	
y_1	$\frac{1}{6}$	1	0	$\frac{5}{6}$	$-\frac{5}{6}$	$\frac{23}{6}$
y_4	$-\frac{1}{6}$	0	-1	$\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{6}$
y_7	[$\frac{2}{3}$]	-1	0	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{13}{3}$
-C'	$-\frac{1}{6}$	1	0	$M-\frac{29}{6}$	$\frac{29}{6}$	$-\frac{119}{6}$

	y_7	y_2	y_3	y_0	y_6	
y_1	$-\frac{1}{4}$	$\frac{5}{4}$	0	$\frac{3}{4}$	$-\frac{3}{4}$	$\frac{11}{4}$
y_4	$\frac{1}{4}$	$-\frac{1}{4}$	-1	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{5}{4}$
y_5	$\frac{3}{2}$	$-\frac{3}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{13}{2}$
-C'	$\frac{1}{4}$	$\frac{3}{4}$	0	$M-\frac{57}{12}$	$\frac{19}{4}$	$-\frac{75}{4}$

The optimal solution is $(\frac{11}{4}, 0, 0, \frac{5}{4}, \frac{13}{2}, 0, 0)$, for which $C' = \frac{75}{4} = C$.

□

12-7. An automobile manufacturer produces two basic car models A and B. They are sold to car dealers at a profit of $200 per car A and $100 per car B. A requires, on the average, 150 man-hours for assembly, 50 man-hours for painting and finishing, and 10 man-hours for checking out and testing. B averages 60 man-hours for assembly, 40 man-hours for painting and finishing, and 20 man-hours for checkout and testing. During each production run, there are 30,000 man-hours available in the assembly shops, 13,000 man-hours in the painting and finishing shops, and 5,000 man-hours in the checking and testing division. Use the simplex

method to determine the number of each model the manufacturer should plan to produce so as to realize the greatest possible profit from each production run.

Let x_1 = number of model A cars

x_2 = " of " B "

Maximize $C = 200\,x_1 + 100\,x_2$

subject to

$$\begin{aligned} 150\,x_1 + 60\,x_2 &\le 30{,}000 \\ 50\,x_1 + 40\,x_2 &\le 13{,}000 \\ 10\,x_1 + 20\,x_2 &\le 5{,}000 \end{aligned}$$

$$x_1,\ x_2 \ge 0$$

Adding slack variables

$$\begin{aligned} 150\,x_1 + 60\,x_2 + x_3 &= 30{,}000 \\ 50\,x_1 + 40\,x_2 + x_4 &= 13{,}000 \\ 10\,x_1 + 20\,x_2 + x_5 &= 5{,}000 \end{aligned}$$

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5 \ge 0$$

A basic feasible solution is (0,0,30 000, 13 000, 5 000) with x_1, x_2 as basic variables.

Applying the simplex method

	x_1	x_2	
x_3	$\boxed{150}$	60	30,000
x_4	50	40	13,000
x_5	10	20	5,000
C	-200	-100	0

	x_3	x_2	
x_1	$\frac{1}{150}$	$\frac{2}{5}$	200
x_4	$-\frac{1}{3}$	$\boxed{20}$	3,000
x_5	$-\frac{1}{15}$	16	3,000
C	$\frac{4}{3}$	-20	40,000

	x_3	x_4	
x_1	$\frac{1}{75}$	$-\frac{1}{50}$	140
x_2	$-\frac{1}{60}$	$\frac{1}{20}$	150
x_5	$\frac{1}{5}$	$-\frac{4}{5}$	600
C	1	1	43,000

The optimal solution is (140, 150, 0, 0, 600) for which

$$C = 43{,}000$$

□

12-8. The values of a continuous function f(x) at x = 0, 1, 2, and 3 are

x	0	1	2	3
f(x)	2	-2	1	4

f(x) may be approximated by a second-degree polynomial

$$g(x) = a_0 + a_1x + a_2x^2$$

such that the maximal value of the approximation error $|g(x) - f(x)|$ at x = 0, 1, 2, and 3 is minimized.

(a) Formulate this minimization problem as a linear programming problem.

(b) Formulate the dual problem.

(a) $|g(0) - f(0)| = |a_0 - 2|$

$|g(1) - f(1)| = |a_0 + a_1 + a_2 + 2|$

$|g(2) - f(2)| = |a_0 + 2a_1 + 4a_2 - 1|$

$|g(3) - f(3)| = |a_0 + 3a_1 + 9a_2 - 4|$

Let $u = \max (|a_0 - 2|, |a_0 + a_1 + a_2 + 2|, |a_0 + 2a_1 + 4a_2 - 1|, |a_0 + 3a_1 + 9a_2 - 4|)$

Then

$-u \le a_0 - 2 \le u$

$-u \le a_0 + a_1 + a_2 + 2 \le u$

$-u \le a_0 + 2a_1 + 4a_2 - 1 \le u$

$-u \le a_0 + 3a_1 + 9a_2 - 4 \le u$

The variables a_0, a_1, a_2 are unrestricted in sign.

Let

$a_0 = a_0' - a_0''$ where $a_0', a_0'' \ge 0$

$a_1 = a_1' - a_1''$ where $a_1', a_1'' \ge 0$

$a_2 = a_2' - a_2''$ where $a_2', a_2'' \ge 0$

Then the problem may be expressed as the following linear programming problem.

Minimize

$$C = u$$

subject to

$$\begin{aligned}
-a_0' + a_0'' &+ u \ge -2\\
a_0' - a_0'' &+ u \ge 2\\
-a_0' + a_0'' - a_1' + a_1'' - a_2' + a_2'' &+ u \ge 2\\
a_0' - a_0'' + a_1' - a_1'' + a_2' - a_2'' &+ u \ge -2\\
-a_0' + a_0'' - 2a_1' + 2a_1'' - 4a_2' + 4a_2'' &+ u \ge -1\\
a_0' - a_0'' + 2a_1' - 2a_1'' + 4a_2' - 4a_2'' &+ u \ge 1\\
-a_0' + a_0'' - 3a_1' + 3a_1'' - 9a_2' + 9a_2'' &+ u \ge -4\\
a_0' - a_0'' + 3a_1' - 3a_1'' + 9a_2' - 9a_2'' &+ u \ge 4\\
a_0', a_0'', a_1', a_1'', a_2', a_2'', u &\ge 0
\end{aligned}$$

(b) The dual problem is

Maximize $C' = -2y_1 + 2y_2 + 2y_3 - 2y_4 - y_5 + y_6 - 4y_7 + 4y_8$

subject to

$$\begin{aligned}
-y_1 + y_2 - y_3 + y_4 - y_5 + y_6 - y_7 + y_8 &\le 0 \\
y_1 - y_2 + y_3 - y_4 + y_5 - y_6 + y_7 - y_8 &\le 0 \\
-y_3 + y_4 - 2y_5 + 2y_6 - 3y_7 + 3y_8 &\le 0 \\
y_3 - y_4 + 2y_5 - 2y_6 + 3y_7 - 3y_8 &\le 0 \\
-y_3 + y_4 - 4y_5 + 4y_6 - 9y_7 + 9y_8 &\le 0 \\
y_3 - y_4 + 4y_5 - 4y_6 + 9y_7 - 9y_8 &\le 0 \\
y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 &\le 1
\end{aligned}$$

$$y_1, y_2, \cdots, y_8 \ge 0.$$

The constraints may be more simply written as

$$\begin{aligned}
y_1 - y_2 + y_3 - y_4 + y_5 - y_6 + y_7 - y_8 &= 0 \\
y_3 - y_4 + 2y_5 - 2y_6 + 3y_7 - 3y_8 &= 0 \\
y_3 - y_4 + 4y_5 - 4y_6 + 9y_7 - 9y_8 &= 0 \\
y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 &\le 1
\end{aligned}$$

$$y_1, y_2, \cdots, y_8 \ge 0.$$

Corresponding to each pair of variables a_i', a_i'', $i = 0, 1, 2$, in the primal problem, there is an equality constraint in the dual problem.

□

12-9. A pound of liver contains 0.1 units of protein and 0.04 units of iron. A pound of beef contains 0.3 units of protein and 0.016 units of iron. Both liver and beef cost \$1/lb. The minimal daily requirement for a student is 0.5 units of protein and 0.048 units of iron.

(a) What quantities of liver and beef should a student consume daily to meet his minimal requirement at the smallest total cost?

(b) Formultate the dual of the linear programming problem formulated in part (a).

(a) Let x_1 and x_2 be the number of pounds of liver and beef, respectively.

Minimize $C = x_1 + x_2$

subject to

$$\begin{aligned}
x_1 + 3x_2 &\ge 5 \\
5x_1 + 2x_2 &\ge 6 \\
x_1, x_2 &\ge 0
\end{aligned}$$

Adding slack variables

$$x_1 + 3x_2 - x_3 = 5$$
$$5x_1 + 2x_2 - x_4 = 6$$
$$x_1, x_2, x_3, x_4 \geq 0$$

A basic feasible solution is $(\frac{8}{13}, \frac{19}{13}, 0, 0)$. The corresponding tableau

	x_3	x_4	
x_1	$\frac{2}{13}$	$-\frac{3}{13}$	$\frac{8}{13}$
x_2	$-\frac{5}{13}$	$\frac{1}{13}$	$\frac{19}{13}$
$-C$	$\frac{3}{13}$	$\frac{2}{13}$	$-\frac{27}{13}$

indicates that this is an optimal solution, for which

$$C = \frac{27}{13} .$$

(b) The dual problem is

Maximize $C' = 5y_1 + 6y_2$

subject to
$$y_1 + 5y_2 \leq 1$$
$$3y_1 + 2y_2 \leq 1$$
$$y_1, y_2 \geq 0$$

Adding slack variables

$$y_1 + 5y_2 + y_3 = 1$$
$$3y_1 + 2y_2 + y_4 = 1$$
$$y_1, y_2, y_3, y_4 \geq 0$$

A basic feasible solution is (0, 0, 1, 1). Applying the simplex method

	y_1	y_2	
y_3	1	5	1
y_4	$\boxed{3}$	2	1
C'	-5	-6	0

	y_4	y_2	
y_3	$-\frac{1}{3}$	$\boxed{\frac{13}{3}}$	$\frac{2}{3}$
y_1	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$
C'	$\frac{5}{3}$	$-\frac{8}{3}$	$\frac{5}{3}$

	y_4	y_3	
y_2	$-\frac{1}{13}$	$\frac{3}{13}$	$\frac{2}{13}$
y_1	$\frac{5}{13}$	$-\frac{2}{13}$	$\frac{3}{13}$
C'	$\frac{19}{13}$	$\frac{8}{13}$	$\frac{27}{13}$

The optimal solution is $(\frac{3}{13}, \frac{2}{13}, 0, 0)$, for which

$$C' = \frac{27}{13} = C.$$

□

12-10. Prove that if the kth constraint in a linear programming problem is an equality, then the corresponding variable y_k in the dual problem is unrestricted in sign, and conversely.

If the k^{th} constraint of a primal problem is an equality,

$$a_{k1}\, x_1 + a_{k2}\, x_2 + \cdots + a_{kr}\, x_r = b_k$$

it can be replaced by the two inequalities

$$a_{k1}\, x_1 + a_{k2}\, x_2 + \cdots + a_{kr}\, x_r \leq b_k$$
$$-a_{k1}\, x_1 - a_{k2}\, x_2 - \cdots - a_{kr}\, x_r \leq -b_k$$

Corresponding to these two inequality constraints in the modified primal problem are two non-negative variables y_k' and y_k'' in its dual problem. Every linear function found in this dual problem depends on the difference

$$y_k' - y_k'' .$$

In the i^{th} constraint, $i = 1, 2, \cdots, r$, this dependence is

$$a_{ki}\, y_k' - a_{ki}\, y_k'' = a_{ki}\, (y_k' - y_k'')$$

while in the objective function, it is

$$b_k\, y_k' - b_k\, y_k'' = b_k\, (y_k' - y_k'')$$

Thus, the substitution

$$y_k' - y_k'' = y_k$$

can be made in this dual problem. The new variable y_k is, in general, unrestricted in sign, since y_k' and y_k'' are arbitrary non-negative variables. Associating the variable y_k with the equality k^{th} constraint in the original primal problem establishes the desired result.

Conversely, if a variable y_k of a dual problem is unrestricted in sign, it can be replaced by the difference

$$y_k' - y_k''$$

of two non-negative variables.

The dependence on y_k' and y_k'' in the objective function is of the form

$$b_k\, (y_k' - y_k'')$$

and in the i^{th} constraint, $i = 1, 2, \cdots, r$, is of the form

$$a_{ki}\, (y_k' - y_k'')$$

Corresponding to y_k' and y_k'' in this dual problem, are two inequality constraints in the primal problem

$$a_{k1} x_1 + a_{k2} x_2 + \cdots + a_{kr} x_r \leq (\geq)\ b_k$$

$$-a_{k1} x_1 - a_{k2} x_2 - \cdots - a_{kr} x_r \leq (\geq)\ -b_k$$

which may be combined to give the equality constraint

$$a_{k1} x_1 + a_{k2} x_2 + \cdots + a_{kr} x_r = b_k.$$

□

12-11. Describe how the simplex method may be applied directly to a linear programming problem in which the objective function is to be minimized, without reformulation as a maximization problem.

Express the objective function in terms of the non-basic variables of a basic feasible solution.

$$C + v_{0,i_1} x_{i_1} + v_{0;i_2} x_{i_2} + \cdots + v_{0,i_r} x_{i_r} = w_0$$

If, for some $j = 1, 2, \cdots, r$, v_{0,i_j} is positive, then the value of C can be decreased by making x_{i_j} a basic variable. If $v_{0,i_j} \leq 0$, for all $j = 1, 2, \cdots, r$, then an optimum (minimum) solution has been found.

The rules for determining which basic variable should be made non-basic and the rules for transforming the tableau do not change.

□

12-12. When there is no obvious basic feasible solution to a linear programming problem, one can be found by introducing artificial variables, as discussed in Sec. 12-6. Another method for finding a basic feasible solution is presented in this problem. Consider the example used in Sec. 12-6 and repeated below.

Maximize the function $C = 5x_1 + 3x_2$ subject to the constraints

$$-4x_1 + 5x_2 \leq -10$$

$$5x_1 + 2x_2 \leq 10$$

$$3x_1 + 8x_2 \leq 12$$

$$x_1 \geq 0 \quad x_2 \geq 0$$

(a) Use the simplex method to minimize the function $C' = x_0$ subject to the constraints

$$\begin{aligned} -4x_1 - 5x_2 - x_0 &\le -10 \\ 5x_1 + 2x_2 &\le 10 \\ 3x_1 + 8x_2 &\le 12 \end{aligned}$$

$$x_1 \ge 0 \qquad x_2 \ge 0 \qquad x_0 \ge 0$$

What is the significance of the optimal solution to this problem?

(b) Use the optimal solution obtained in part (a) to solve the original maximization problem.

(c) State a general procedure for finding a basic feasible solution to a linear programming problem. Under what condition will a linear programming problem possess no feasible solution?

(a) Adding slack variables

$$\begin{aligned} -4x_1 - 5x_2 + x_3 - x_0 &= -10 \\ 5x_1 + 2x_2 + x_4 &= 10 \\ 3x_1 + 8x_2 + x_5 &= 12 \end{aligned}$$

$$x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5 \ge 0$$

A basic feasible solution is $(x_0,x_1,x_2,x_3,x_4,x_5) = (10,0,0,0,10,12)$. Applying the simplex method

	x_1	x_2	x_3	
x_0	4	5	-1	10
x_4	**5**	2	0	10
x_5	3	8	0	12
$-C'$	-4	-5	1	-10

	x_4	x_2	x_3	
x_0	$-\frac{4}{5}$	$\frac{17}{5}$	-1	2
x_1	$\frac{1}{5}$	$\frac{2}{5}$	0	2
x_5	$-\frac{3}{5}$	$\frac{34}{5}$	0	6
$-C'$	$\frac{4}{5}$	$-\frac{17}{5}$	1	-2

	x_4	x_0	x_3	
x_2	$-\frac{4}{17}$	$\frac{5}{17}$	$-\frac{5}{17}$	$\frac{10}{17}$
x_1	$\frac{5}{17}$	$-\frac{2}{17}$	$\frac{2}{17}$	$\frac{30}{17}$
x_5	1	-2	2	2
$-C'$	0	1	0	0

An optimal solution is $(x_0,x_1,x_2,x_3,x_4,x_5) = (0, \frac{30}{17}, \frac{10}{17}, 0, 0, 2)$. For which $C' = 0$.

$(x_1,x_2,x_3,x_4,x_5) = (\frac{30}{17}, \frac{10}{17}, 0, 0, 2)$ is a basic feasible solution to the original problem.

(b) Adding slack variables to the original problem,

$$\begin{aligned} -4x_1 - 5x_2 + x_3 &= -10 \\ 5x_1 + 2x_2 + x_4 &= 10 \\ 3x_1 + 8x_2 + x_5 &= 12 \end{aligned}$$

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5 \ge 0.$$

From part (a), $(x_1,x_2,x_3,x_4,x_5) = (\frac{30}{17}, \frac{10}{17}, 0, 0, 2)$ is a basic feasible solution. Applying the simplex method

	x_3	x_4	
x_1	$\frac{2}{17}$	$\frac{5}{17}$	$\frac{30}{17}$
x_2	$-\frac{5}{17}$	$-\frac{4}{17}$	$\frac{10}{17}$
x_5	$\boxed{2}$	1	2
C	$-\frac{5}{17}$	$\frac{13}{17}$	$\frac{180}{17}$

	x_5	x_4	
x_1	$-\frac{1}{17}$	$\frac{4}{17}$	$\frac{28}{17}$
x_2	$\frac{5}{34}$	$-\frac{3}{34}$	$\frac{15}{17}$
x_3	$\frac{1}{2}$	$\frac{1}{2}$	1
C	$\frac{5}{34}$	$\frac{31}{34}$	$\frac{185}{17}$

The optimal solution is $(x_1,x_2,x_3,x_4,x_5) = (\frac{28}{17}, \frac{15}{17}, 1, 0, 0)$, for which $C = \frac{185}{17}$.

(c) Add artificial variables to the constraints in such a way as to make a basic feasible solution for these augmented constraints obvious. Let C' be the sum of the artificial variables. Consider the new problem of minimizing C' subject to the augmented constraints.

In an optimal basic feasible solution to this new problem, if the minimum value for C' is 0, then each of the artificial variables has value 0. The values assigned to the remaining variables constitutes a basic feasible solution to the original problem.

In an optimal basic feasible solution to the new problem, if the minimum value for C' is greater than 0, then there can be no feasible solution to the original problem. If there were, then augmenting this feasible solution with values of 0 for the artificial variables would give a feasible solution to the new problem for which C' = 0, a contradiction.

□

12-13. In this problem another method of solving linear programming problems, known as the dual simplex method, is studied. In the dual simplex method, a sequence of basic solutions is generated, each having the property that the coefficients in the bottom row of the corresponding tableau are all positive. The method terminates when a basic feasible solution is found. Consider the problem of maximizing the function $C = -x_1 - 2x_2 - 5x_3$ subject to the constraints

$$\begin{aligned} 3x_1 - 5x_2 + 2x_3 &\le -3 \\ x_1 + x_2 - 3x_3 &\le 1 \\ 2x_1 - x_2 - 3x_3 &\le -2 \end{aligned}$$

$$x_1 \ge 0 \qquad x_2 \ge 0 \qquad x_3 \ge 0$$

Adding slack variables, these constraints become

$$\begin{aligned} 3x_1 - 5x_2 + 2x_3 + x_4 &= -3 \\ x_1 + x_2 - 3x_3 + x_5 &= 1 \\ 2x_1 - x_2 - 3x_3 + x_6 &= -2 \end{aligned}$$

It is clear that $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = -3$, $x_5 = 1$, and $x_6 = -2$ constitutes a basic solution, although it is not a feasible one.

(a) Set up the tableau corresponding to this basic solution.

(b) Since both x_4 and x_6 assume negative values, one should be made a non-basic variable. If x_4 is made a non-basic variable, which variable should be made a basic variable? Set up the new tableau.

(c) At this point x_6 should be made a non-basic variable. Which variable should be made a basic variable? Why? Complete the problem.

(d) State in general terms the steps necessary to carry out the dual simplex method.

(a)

	x_1	x_2	x_3	
x_4	3	[-5]	2	-3
x_5	1	1	-3	1
x_6	2	-1	-3	-2
C	1	2	5	0

(b) The constraint corresponding to the first row of the tableau in part (a) is

$$x_4 + 3x_1 - 5x_2 + 2x_3 = -3.$$

In making x_4 a non-basic variable, x_2 is the only non-basic variable which can be made a basic variable having positive value, and should be chosen. With the pivot identified, the tableau may be transformed according to the rules of the simplex method to obtain the following new tableau

	x_1	x_4	x_3	
x_2	$-\frac{3}{5}$	$-\frac{1}{5}$	$-\frac{2}{5}$	$\frac{3}{5}$
x_5	$\frac{8}{5}$	$\frac{1}{5}$	$-\frac{13}{5}$	$\frac{2}{5}$
x_6	$\frac{7}{5}$	$-\frac{1}{5}$	$\left[-\frac{17}{5}\right]$	$-\frac{7}{5}$
C	$\frac{11}{5}$	$\frac{2}{5}$	$\frac{29}{5}$	$-\frac{6}{5}$

(c) The constraint corresponding to the third row of the tableau in part (b) is

$$x_6 + \frac{7}{5} x_1 - \frac{1}{5} x_4 - \frac{17}{5} x_3 = -\frac{7}{5}$$

In making x_6 a non-basic variable, either x_4 or x_3 can be made a basic variable having positive positive value. However, choosing x_4 would give a negative coefficient in the bottom row of the new tableau in the x_3 column. This is a consequence of the fact that

$$\frac{\frac{2}{5}}{-\frac{1}{5}} < \frac{\frac{29}{5}}{-\frac{17}{5}}$$

as will be seen in part (d) below.

Choosing x_3 to be made a basic variable, the following final tableau results.

	x_1	x_4	x_6	
x_2	$-\frac{13}{17}$	$-\frac{3}{17}$	$-\frac{2}{17}$	$\frac{13}{17}$
x_5	$\frac{9}{17}$	$\frac{6}{17}$	$-\frac{13}{17}$	$\frac{25}{17}$
x_3	$-\frac{7}{17}$	$\frac{1}{17}$	$-\frac{5}{17}$	$\frac{7}{17}$
C	$\frac{78}{17}$	$\frac{1}{17}$	$\frac{29}{17}$	$-\frac{61}{17}$

The optimal solution is $(0, \frac{13}{17}, \frac{7}{17}, 0, \frac{25}{17}, 0)$, for which $C = -\frac{61}{17}$.

(d) Consider a maximization problem.

Start with the tableau corresponding to a basic solution (not necessarily feasible) with bottom row coefficients all positive. (The dual simplex method can be applied when some of these coefficients are zero, a degenerate case, but it raises the problem of cycling. Although there is a method for dealing with this, it will be not considered here.)

	x_{m+1}	x_{m+2}	$\cdots$	x_{m+j}	$\cdots$	x_{m+r}	
x_1	$v_{1,m+1}$	$v_{1,m+2}$	$\cdots$	$v_{1,m+j}$	$\cdots$	$v_{1,m+r}$	w_1
x_2	$v_{2,m+1}$	$v_{2,m+2}$	$\cdots$	$v_{2,m+j}$	$\cdots$	$v_{2,m+r}$	w_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_k	$v_{k,m+1}$	$v_{k,m+2}$	$\cdots$	$v_{k,m+j}$	$\cdots$	$v_{k,m+r}$	w_k
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_m	$v_{m,m+1}$	$v_{m,m+2}$	$\cdots$	$v_{m,m+j}$	$\cdots$	$v_{m,m+r}$	w_m
C	$v_{0,m+1}$	$v_{0,m+2}$	$\cdots$	$v_{0,m+j}$	$\cdots$	$v_{0,m+r}$	w_0

$$v_{0,m+1}, v_{0,m+2}, \cdots, v_{0,m+j}, \cdots, v_{0,m+r} > 0$$

If some $w_k < 0$, then the basic variable x_k can be made into a non-basic variable by increasing its value to 0. To determine which of the variables $x_{m+1}, x_{m+2}, \cdots, x_{m+r}$ should become a basic variable, compute the ratios:

$$\frac{v_{0,m+1}}{v_{k,m+1}}, \frac{v_{0,m+2}}{v_{k,m+2}}, \cdots, \frac{v_{0,m+r}}{v_{k,m+r}}$$

Suppose that the ratio $\frac{v_{0,m+j}}{v_{k,m+j}}$ is the largest of all the <u>negative</u> quantities obtained. Then x_{m+j} should be made a basic variable. That is, the tableau entry $v_{k,m+j}$ is the pivot. The rules for transforming the tableau, having identified the pivot, are the same as in the simplex method.

If primes denote new tableau entries after transformation, then

$$v'_{0,m+1}, v'_{0,m+2}, \cdots, v'_{0,m+r} \geq 0$$

Otherwise, for some q

$$v'_{0,m+q} < 0 \qquad q \neq j$$

$$v'_{0,m+q} = v_{0,m+q} - v_{0,m+j} \cdot \frac{v_{k,m+q}}{v_{k,m+j}} < 0$$

$$v_{0,m+q} < v_{0,m+j} \cdot \frac{v_{k,m+q}}{v_{k,m+j}}$$

If $v_{k,m+q} = 0$, then $v_{0,m+q} < 0$, a contradiction.

If $v_{k,m+q} > 0$, then

$$0 < \frac{v_{0,m+q}}{v_{k,m+q}} < \frac{v_{0,m+j}}{v_{k,m+j}} < 0, \text{ a contradiction.}$$

If $v_{k,m+q} < 0$, then

$$0 > \frac{v_{0,m+q}}{v_{k,m+q}} > \frac{v_{0,m+j}}{v_{k,m+j}}$$

which contradicts the fact that $\frac{v_{0,m+j}}{v_{k,m+j}}$ was the largest among the computed ratios having negative value.

Thus, the sucessor basic solution has the property that all coefficients in the bottom row of its corresponding tableau are non-negative. They will all be positive unless, for some $q \neq j$,

$$\frac{v_{0,m+q}}{v_{k,m+q}} = \frac{v_{0,m+j}}{v_{k,m+j}}$$

The situation may arise in which

$$w_k < 0 \quad \text{and} \quad v_{k,m+1}, v_{k,m+2}, \cdots, v_{k,m+r} \geq 0$$

Then, none of the ratios

$$\frac{v_{0,m+1}}{v_{k,m+1}}, \frac{v_{0,m+2}}{v_{k,m+2}}, \cdots, \frac{v_{0,m+r}}{v_{k,m+r}}$$

is negative and no choice of a non-basic variable to be made basic is possible. In this case, it may be seen from the equation

$$x_k + v_{k,m+1} x_{m+1} + v_{k,m+2} x_{m+2} + \cdots + v_{k,m+r} x_{m+r} = w_k < 0$$

that there is no feasible solution satisfying this constraint equation. Thus the problem has no feasible solution.

The value of C for each successive basic (non-feasible) solution generated by the dual simplex method decreases, until an optimal basic feasible solution is obtained. This follows from the fact that

$$w_0' = w_0 - v_{0,m+j} \frac{w_k}{v_{k,m+j}}$$

and, since

$$v_{0,m+j} > 0, \quad w_k < 0, \quad v_{k,m+j} < 0$$

then

$$v_{0,m+j} \frac{w_k}{v_{k,m+j}} > 0.$$

□

12-14. To start the dual simplex method, a basic solution is needed for which the coefficients in the bottom row of the corresponding tableau are all positive. Consider the problem of maximizing the function $C = x_1 + 4x_2 + 2x_3$ subject to the constraints

$$3x_1 + x_2 + 4x_3 \leq 2$$
$$x_1 + 2x_2 + x_3 \leq 5$$
$$x_1 \geq 0 \quad x_2 \geq 0 \quad x_3 \geq 0$$

Since there is no obvious basic solution that satisfies the condition, the additional constraint

$$x_1 + x_2 + x_3 \leq M$$

is introduced, where M is a very large number. Clearly, this does not change the problem. Adding slack variables, the constraints become

$$
\begin{aligned}
3x_1 + x_2 + 4x_3 + x_4 &= 2 \\
x_1 + 2x_2 + x_3 + x_5 &= 5 \\
x_1 + x_2 + x_3 + x_6 &= M
\end{aligned}
$$

A basic solution in which x_6 and two of the three variables x_1, x_2, and x_3 are non-basic variables will have all positive coefficients in the bottom row of the corresponding tableau.

(a) Find this basic solution.

(b) Solve the problem using the dual simplex method.

(c) Maximize the objective function $C'' = x_1 - 4x_2 + 2x_3$ under the same set of constraints using the dual simplex method.

(d) State in general terms the steps necessary to find a basic solution for starting the dual simplex method.

(a) Choosing x_1, x_3 and x_6 to be non-basic variables gives the desired basic (but not feasible) solution, the corresponding tableau being

	x_1	x_3	x_6	
x_2	1	1	1	M
x_4	2	3	-1	2-M
x_5	-1	-1	-2	5-2M
C	3	2	4	4M

(b) Applying the dual simplex method

	x_1	x_3	x_6	
x_2	1	1	1	M
x_4	2	3	[-1]	2-M
x_5	-1	-1	-2	5-2M
C	3	2	4	4M

	x_1	x_3	x_4	
x_2	3	4	1	2
x_6	-2	-3	-1	M-2
x_5	-5	-7	-2	1
C	11	14	4	8

The optimal solution to the modified problem is (0,2,0,0,1,M-2) for which C = 8.

The optimal solution to the original problem is (0,2,0,0,1) for which C = 8.

(c) Choosing x_1, x_2 and x_6 to be non-basic variables gives a basic (but not feasible) solution of the desired kind, the corresponding tableau being

	x_1	x_2	x_6	
x_3	1	1	1	M
x_4	-1	-3	-4	2-4M
x_5	0	1	[-1]	5-M
C'	1	6	2	2M

With the pivot chosen as indicated, the dual simplex method generates the following tableaux.

	x_1	x_2	x_5	
x_3	1	2	1	5
x_4	-1	-7	[-4]	-18
x_6	0	-1	-1	M-5
C'	1	8	2	10

	x_1	x_2	x_4	
x_3	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
x_5	$\frac{1}{4}$	$\frac{7}{4}$	$-\frac{1}{4}$	$\frac{9}{2}$
x_6	$\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$M-\frac{1}{2}$
C'	$\frac{1}{2}$	$\frac{9}{2}$	$\frac{1}{2}$	1

The optimal solution to the modified problem is $(0, 0, \frac{1}{2}, 0, \frac{9}{2}, M-\frac{1}{2})$ for which C' = 1.

The optimal solution to the un-modified problem is $(0, 0, \frac{1}{2}, 0, \frac{9}{2})$ for which C' = 1.

(d) Consider a maximization problem in which there are r variables and m constraints. Let the objective function be

$$C = c_1x_1 + c_2x_2 + \cdots + c_rx_r \qquad [1]$$

and let c_i be the largest coefficient appearing in the right hand side of [1].

If $c_i \leq 0$, then choosing $x_1, x_2, \cdots, x_r$ to be non-basic gives a basic solution suitable for starting the dual simplex method.

If $c_i > 0$, then add the constraint

$$x_1 + x_2 + \cdots + x_r \leq M$$

where M is a large positive constant. When slack variables are added, this new constraint becomes

$$x_1 + x_2 + \cdots + x_r + x_{m+r+1} = M \qquad [2]$$

Solving [2] for x_i

$$x_i = M - x_1 - x_2 - \cdots - x_{i-1} - x_{i+1} - \cdots - x_r - x_{m+r+1}$$

and substituting in [1]

$$C + (c_i-c_1)x_1 + (c_i-c_2)x_2 + \cdots + (c_i-c_{i-1})x_{i-1} + (c_i-c_{i+1})x_{i+1} + \cdots +$$
$$(c_i-c_r)x_r + c_i\, x_{m+4+1} = c_i\, M \qquad [3]$$

By the choice of c_i, all coefficients in [3] are non-negative. Choosing $x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_r, x_{m+r+1}$ to be non-basic gives a basic solution to the modified problem suitable for starting the dual simplex method. Once an optimal solution to the modified problem has been found, it becomes an optimal solution for the original problem when the variable x_{m+r+1} is dropped.

□

13-1. Find the maximal value of the product $x_1 x_2 \cdots x_n$ subject to the constraints

$$x_1 + x_2 + \cdots + x_n = 1$$

$$x_1 \geq 0,\ x_2 \geq 0,\ \cdots,\ x_n \geq 0$$

This may be treated as a multi-stage decision problem in which the quantity 1 is to be divided into n non-negative parts in n steps.

Let $f_k(a)$ denote the maximal value of the product $x_1 x_2 \cdots x_k$, when the quantity a is divided into k non-negative parts $x_1, x_2, \cdots, x_k$.

Then

$$f_1(a) = a$$

$$f_k(a) = \max_{0 \leq x \leq a} [x\ f_{k-1}(a-x)] \qquad [1]$$

Using [1] in an iterative manner

$$f_1(a) = a$$

$$f_2(a) = \max_{0 \leq x \leq a} [x\ f_1(a-x)]$$

$$= \max_{0 \leq x \leq a} [x(a-x)]$$

$$= \left(\frac{a}{2}\right)^2 \qquad \text{(with } x = \frac{a}{2}\text{)}$$

$$f_3(a) = \max_{0 \leq x \leq a} [x\ f_2(a-x)]$$

$$= \max_{0 \leq x \leq a} \left[x \left(\frac{a-x}{2}\right)^2\right]$$

$$= \left(\frac{a}{3}\right)^3 \qquad \text{(with } x = \frac{a}{3}\text{)}$$

In general,

$$f_k(a) = \left(\frac{a}{k}\right)^k$$

which may be verified as follows.

$$\max_{0 \leq x \leq a} [x\ f_{k-1}(a-x)] = \max_{0 \leq x \leq a} \left[x \left(\frac{a-x}{k-1}\right)^{k-1}\right]$$

$$\frac{d}{dx}\left[x \left(\frac{a-x}{k-1}\right)^{k-1}\right] = \frac{1}{(k-1)^{k-1}} (a-x)^{k-2} (a-kx)$$

The derivative has value 0 at $x = a$ and $x = \frac{a}{k}$.

The maximum is attained at $x = \frac{a}{k}$.

$$\max_{0 \le x \le a} [x\, f_{k-1}(a-x)] = \frac{a}{k} \left(\frac{a - \frac{a}{k}}{k-1}\right)^{k-1} = \left(\frac{a}{k}\right)^{k} = f_k(a)$$

Therefore

$$f_n(1) = \left(\frac{1}{n}\right)^{n}$$

The optimal policy is found, by recursive use of [1], to be

$$x_i = \frac{1}{n} \qquad \text{for } i = 1, 2, \cdots, n$$

□

13-2. Minimize the quantity $x_1^2 + x_2^2 + \cdots + x_n^2$ subject to the constraints

$$x_1 + x_2 + \cdots + x_n = 1$$

$$x_1 \ge 0,\ x_2 \ge 0,\ \cdots,\ x_n \ge 0$$

This may be treated as a multi-stage decision problem in which the quantity 1 is to be divided into n non-negative parts in n steps.

Let $f_k(a)$ denote the minimal value of the sum $x_1^2 + x_2^2 + \cdots + x_k^2$, when the quantity a is divided into k non-negative parts $x_1, x_2, \cdots, x_k$.

Then

$$f_1(a) = a^2$$

$$f_k(a) = \min_{0 \le x \le a} [x^2 + f_{k-1}(a-x)] \qquad [1]$$

Using [1] in an iterative manner

$$f_1(a) = a^2$$

$$f_2(a) = \min_{0 \le x \le a} [x^2 + f_1(a-x)]$$

$$= \min_{0 \le x \le a} [x^2 + (a-x)^2]$$

$$= \frac{a^2}{2} \qquad (\text{with } x = \tfrac{a}{2})$$

$$f_3(a) = \min_{0 \le x \le a} [x^2 + f_2(a-x)]$$

$$= \min_{0 \le x \le a} \left[x^2 + \frac{(a-x)^2}{2}\right]$$

$$= \frac{a^2}{3} \qquad (\text{with } x = \tfrac{a}{3})$$

In general,

$$f_k(a) = \frac{a^2}{k}$$

which may be verified as follows.

$$\min_{0 \le x \le a} [x^2 + f_{k-1}(a-x)] = \min_{0 \le x \le a} [x^2 + \frac{(a-x)^2}{k-1}]$$

$$\frac{d}{dx} [x^2 + \frac{(a-x)^2}{k-1}] = \frac{2}{k-1} (kx - a)$$

The derivative has value 0 at $x = \frac{a}{k}$, at which point the minimum is attained.

$$\min_{0 \le x \le a} [x^2 + f_{k-1}(a-x)] = (\frac{a}{k})^2 + \frac{(a - \frac{a}{k})^2}{k-1}$$

$$= \frac{a^2}{k}$$

$$= f_k(a)$$

Therefore

$$f_n(1) = \frac{1}{n}$$

The optimal policy is found, by recursive use of [1], to be

$$x_i = \frac{1}{n} \qquad \text{for } i = 1, 2, \cdots, n$$

□

13-3. Maximize the quantity $x_1y_1 + x_2y_2 + \cdots + x_ny_n$ subject to the constraints

$$x_1 + x_2 + \cdots + x_n = a$$
$$y_1 + y_2 + \cdots + y_n = b$$
$$x_1 \ge 0, x_2 \ge 0, \cdots, x_n \ge 0$$
$$y_1 \ge 0, y_2 \ge 0, \cdots, y_n \ge 0$$

Let $f_k(u,v)$ denote the maximal value of the sum $x_1y_1 + x_2y_2 + \cdots + x_ky_k$ when the quantity u is divided into k non-negative parts $x_1, x_2, \cdots, x_k$, and the quantity v is divided into k non-negative parts $y_1, y_2, \cdots, y_k$.

$$f_1(u,v) = uv$$

$$f_2(u,v) = \max_{\substack{0 \le x \le u \\ 0 \le y \le v}} [xy + f_1(u-x, v-y)]$$

$$= \max_{\substack{0 \le x \le u \\ 0 \le y \le v}} [xy + (u-x)(v-y)]$$

$$= uv \qquad \text{(at } x = 0, y = 0 \text{ or } x = u, y = v\text{)}$$

$$f_3(u,v) = \max_{\substack{0 \le x \le u \\ 0 \le y \le v}} [xy + f_2(u-x, v-y)]$$

$$= \max_{\substack{0 \le x \le u \\ 0 \le y \le v}} [xy + (u-x)(v-y)]$$

$$= uv \qquad \text{(at } x = 0, y = 0 \text{ or } x = u, y = v\text{)}$$

In general

$$f_k(u,v) = \max_{\substack{0 \le x \le u \\ 0 \le y \le v}} [xy + (u-x)(v-y)]$$

$$= uv \qquad \text{(at } x = 0, y = 0 \text{ or } x = u, y = v\text{)}$$

Therefore

$$f_n(a,b) = ab$$

An optimal policy is

$$x_1 = x_2 = \cdots = x_{n-1} = 0, \quad x_n = a$$

$$y_1 = y_2 = \cdots = y_{n-1} = 0, \quad y_n = b$$

□

13-4. A particle is traveling on a plane from a point (x_1, y_1) to another point (x_2, y_2) at a constant speed c. Find the minimal traveling time by formulating the problem as a dynamic programming problem.

This may be considered an infinite stage process. Let f(a,b) denote the minimal traveling time from the point (a,b) to the point (x_2, y_2).

Then

$$f(x_2, y_2) = 0$$

$$f(a,b) = \min_{0 \le \theta \le 2\pi} [dt + f(a + cdt \cos\theta, b + cdt \sin\theta)]$$

The solution to this functional equation is, as expected,

$$f(a,b) = \frac{1}{c} \sqrt{(x_2-a)^2 + (y_2-b)^2}$$

which may be verified as follows.

$$\min_{0 \le \theta \le 2\pi} [dt + f(a + cdt \cos\theta, b + cdt \sin\theta)]$$

$$= \min_{0 \le \theta \le 2\pi} [dt + \frac{1}{c} \sqrt{(x_2 - a - cdt \cos\theta)^2 + (y_2 - b - cdt \sin\theta)^2}]$$

$$= dt + \frac{1}{c} \sqrt{\min_{0 \le \theta \le 2\pi} [(x_2 - a - cdt \cos\theta)^2 + (y_2 - b - cdt \sin\theta)^2]}$$

$$\frac{d}{d\theta} [(x_2 - a - cdt \cos\theta)^2 + (y_2 - b - cdt \sin\theta)^2]$$

$$= -2 (x_2 - a - cdt \cos\theta)\, cdt \sin\theta + 2 (y_2 - b - cdt \sin\theta)\, cdt \cos\theta$$

Setting this derivative equal to 0 and solving for θ,

$$(x_2-a) \sin\theta = (y_2-b) \cos\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y_2-b}{x_2-a}$$

$$\theta = \tan^{-1} \frac{y_2-b}{x_2-a}$$

The minimum is attained at this value of θ.

$$\min_{0 \le \theta \le 2\pi} [dt + f(a + cdt \cos\theta, b + cdt \sin\theta)]$$

$$= dt + \frac{1}{c} \sqrt{((x_2-a) - cdt \frac{(x_2-a)}{\sqrt{(x_2-a)^2 + (y_2-b)^2}})^2 + ((y_2-b) - cdt \frac{(y_2-b)}{\sqrt{(x_2-a)^2 + (y_2-b)^2}})^2}$$

$$= dt + \frac{1}{c} \sqrt{(x_2-a)^2 + (y_2-b)^2 - 2\, cdt \sqrt{(x_2-a)^2 + (y_2-b)^2} + (cdt)^2}$$

$$= dt + \frac{1}{c} \sqrt{(\sqrt{(x_2-a)^2 + (y_2-b)^2} - cdt)^2}$$

$$= dt + \frac{1}{c} (\sqrt{(x_2-a)^2 + (y_2-b)^2} - cdt)$$

$$= \frac{1}{c} \sqrt{(x_2-a)^2 + (y_2-b)^2}$$

$$= f(a,b)$$

Therefore

$$f(x_1, y_1) = \frac{1}{c} \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

The optimal policy is to follow the straight line path connecting (x_1, y_1) and (x_2, y_2)

□

13-5. A chemical processing plant must purify 20,000 lb. of a certain chemical compound in a period of four weeks. There are two purification methods that can be used. The first completely purifies x lbs. of the compound in one week at a cost of $0.60x^2$ dollars. The second second, given x lbs. of the compound to be processed in one week, completely purifies 70 percent of this material and leaves 30 percent to be reprocessed during a subsequent week. It does this at a cost of $0.15x^2$ dollars. Determine an optimal policy for the operation.

Let $f_k(a)$ denote the minimum cost of processing a pounds of the compound during a period of k weeks.

$$f_1(a) = 0.60a^2$$

$$f_2(a) = \min_{0 \le x \le a} [0.60x^2 + 0.15\,(a-x)^2 + f_1(0.30\,(a-x))]$$

$$= \min_{0 \le x \le a} [0.60x^2 + 0.15\,(a-x)^2 + 0.60\,(0.30\,(a-x))^2]$$

$$= \min_{0 \le x \le a} [0.60x^2 + 0.204\,(a-x)^2]$$

$$= 0.1527a^2 \qquad (\text{at } x = 0.254a)$$

$$f_3(a) = \min_{0 \le x \le a} [0.60x^2 + 0.15\,(a-x)^2 + f_2(0.30\,(a-x))]$$

$$= \min_{0 \le x \le a} [0.60x^2 + 0.15\,(a-x)^2 + 0.1527\,(0.30\,(a-x))^2]$$

$$= \min_{0 \le x \le a} [0.60x^2 + 0.1637\,(a-x)^2]$$

$$= 0.1291a^2 \qquad (\text{at } x = 0.214a)$$

$$f_4(a) = \min_{0 \le x \le a} [0.60x^2 + 0.15\,(a-x)^2 + f_3(0.30\,(a-x))]$$

$$= \min_{0 \le x \le a} [0.60x^2 + 0.15\,(a-x)^2 + 0.1291\,(0.30\,(a-x))^2]$$

$$= \min_{0 \le x \le a} [0.60x^2 + 0.1616\,(a-x)^2]$$

$$= 0.1272a^2 \qquad (\text{at } x = 0.212a)$$

The minimum cost for the four-week operation is

$$f_4(20{,}000) = 0.1272\,(20{,}000)^2$$

$$= 50.88 \times 10^6 \quad \text{dollars}$$

The optimal policy is

Week	1	2	3	4
First process	4,220	1,010	284	250
Second process	15,780	3,724	833	0

(pounds of compound)

□

13-6. Problem 5 is to be approximated as an infinite-stage process.

(a) Set up the functional equation.

(b) It is known that the solution to this functional equation is of the form $f(y) = my^2$. Determine the constant m.

(c) Compare the result in part (b) with the result in Problem 5.

(a) Let f(a) denote the minimum cost to process a pounds of the compound.

$$f(a) = \min_{0 \le x \le a} [0.60x^2 + 0.15\ (a-x)^2 + f(0.30\ (a-x))]$$

(b)

$$f(a) = \min_{0 \le x \le a} [0.60x^2 + 0.15\ (a-x)^2 + m(0.30\ (a-x))^2]$$

$$= \min_{0 \le x \le a} [0.60x^2 + (0.15 + 0.09m)(a-x)^2]$$

The minimum occurs at

$$x = \left(\frac{0.15 + 0.09m}{0.75 + 0.09m}\right) a$$

$$f(a) = 0.60\left(\frac{0.15 + 0.09m}{0.75 + 0.09m}\right)^2 a^2 + (0.15 + 0.09m)\ \left(1 - \left(\frac{0.15 + 0.09m}{0.75 + 0.09m}\right)\right)^2 a^2$$

$$f(a) = \frac{0.60a^2\ (0.15 + 0.09m)}{(0.75 + 0.09m)^2}\ [(0.15 + 0.09m) + 0.60]$$

$$= \frac{0.60a^2\ (0.15 + 0.09m)}{0.75 + 0.09m}$$

Equating this to ma^2 and solving for m,

$$0.75m + 0.09m^2 = 0.09 + 0.054m$$

$$0.09m^2 + 0.696m - 0.09 = 0$$

The positive root is

$$m = 0.1272$$

(c) Within the accuracy of these calculations, this result is the same as that obtained for four weeks of operation in Problem 13-5. If $f_5(a)$, $f_6(a)$, $\cdots$, were to be computed in Problem 13-5, with the same degree of accuracy, no further reduction in the value of the objective function would be observed.

□

13-7. A furniture manufacturer has an order for 100 chairs to be delivered in three months. The cost of producing x chairs in the first, second, and third month is $120x$, $1.2x^2$, and $1.5x^2$ dollars, respectively. Determine an optimal policy for the number of chairs to be produced in each month.

Let $f_k(a)$ denote the minimum cost of producing a chairs during a period of k months.

$$f_1(a) = 1.5a^2$$

$$f_2(a) = \min_{0 \le x \le a} [1.2x^2 + f_1(a-x)]$$

$$= \min_{0 \le x \le a} [1.2x^2 + 1.5(a-x)^2]$$

$$= \frac{2}{3}a^2 \qquad (\text{at } x = \frac{5}{9}a)$$

$$f_3(a) = \min_{0 \le x \le a} [120x + f_2(a-x)]$$

$$= \min_{0 \le x \le a} [120x + \frac{2}{3}(a-x)^2]$$

$$= 120a - 5400 \qquad (\text{at } x = a - 90)$$

The minimum production cost is

$$f_3(100) = 120(100) - 5400 = 6600 \text{ dollars.}$$

The optimal production schedule is

Month	1	2	3
Chairs	10	50	40

□

13-8. Two types of machines, I and II, are available to make two kinds of products, A and B, in a five-week period. Machines are assigned to the products on a weekly basis. The weekly attrition rates for the machines, according to the product assigned, are

	Product A	Product B
Type I machines	30%	50%
Type II machines	50%	80%

A machine producing product A yields a profit of \$300 in a week, and a machine producing product B yields a profit of \$400 in a week. Determine an optimal policy for the operation.

Let $f_k(a,b)$ be the maximum profit realizable with a type I machines and b type II machines, during a period of k weeks.

$$f_1(a,b) = \max_{\substack{0 \le x \le a \\ 0 \le y \le b}} [300x + 400\,(a-x) + 300y + 400\,(b-y)]$$

$$= 400a + 400b \qquad (\text{at } x = 0,\ y = 0)$$

$$f_2(a,b) = \max_{\substack{0 \le x \le a \\ 0 \le y \le b}} [300x + 400\,(a-x) + 300y + 400\,(b-y) + f_1(0.7x + 0.5\,(a-x),\ 0.5y + 0.2\,(b-y))]$$

$$= \max_{\substack{0 \le x \le a \\ 0 \le y \le b}} [-20x + 600a + 20y + 480b]$$

$$= 600a + 500b \qquad (\text{at } x = 0,\ y = b)$$

$$f_3(a,b) = \max_{\substack{0 \le x \le a \\ 0 \le y \le b}} [300x + 400\,(a-x) + 300y + 400\,(b-y) + f_2(0.7x + 0.5\,(a-x),\ 0.5y + 0.2\,(b-y))]$$

$$= \max_{\substack{0 \le x \le a \\ 0 \le y \le b}} [20x + 700a + 50y + 500b]$$

$$= 720a + 550b \qquad (\text{at } x = a,\ y = b)$$

$$f_4(a,b) = \max_{\substack{0 \le x \le a \\ 0 \le y \le b}} [300x + 400\,(a-x) + 300y + 400\,(b-y) + f_3(0.7x + 0.5\,(a-x),\ 0.5y + 0.2\,(b-y))]$$

$$= \max_{\substack{0 \le x \le a \\ 0 \le y \le b}} [44x + 760a + 65y + 510b]$$

$$= 804a + 575b \qquad (\text{at } x = a,\ y = b)$$

$$f_5(a,b) = \max_{\substack{0 \le x \le a \\ 0 \le y \le b}} [300x + 400\,(a-x) + 300y + 400\,(b-y) + f_4(0.7x + 0.5\,(a-x),\ 0.5y + 0.2\,(b-y))]$$

$$= \max_{\substack{0 \le x \le a \\ 0 \le y \le b}} [60.8x + 802a + 72.5y + 515b]$$

$$= 862.8a + 587.5b \qquad (\text{at } x = a,\ y = b)$$

Thus, if there are a and b type I and II machines available, respectively, the maximum profit from 5 weeks of operation is

$$f_5(a,b) = 862.8a + 587.5b \quad \text{dollars}$$

The optimal policy is

Week		1		2		3		4		5	
Product		A	B	A	B	A	B	A	B	A	B
Machine	Type I	a	0	$0.7a$	0	$(0.7)^2a$	0	0	$(0.7)^3a$	0	$0.5\,(0.7)^3a$
	Type II	b	0	$0.5b$	0	$(0.5)^2b$	0	$(0.5)^3b$	0	0	$(0.5)^4b$

13-9. Nineteen years of material are available for making three kinds of dresses, which require 2, 3, and 5 yds. of material, respectively. The profit from selling a dress of each kind is \$1.5, \$2.5, and \$4, respectively.

(a) Find all optimal ways of allocating the material to the three kinds of dresses so that the total profit is maximized.

(b) Find a way of allocating the material so that the total profit is the largest integer number of dollars.

This may be treated as a 3-stage process in which dresses of the first, second and third kinds are made in that order.

Let $\begin{pmatrix} f_1(a) \\ f_2(a) \\ f_3(a) \end{pmatrix}$ denote the maximum realizable profit, using a yards of material, from making dresses of the $\begin{pmatrix} \text{third} \\ \text{second, third} \\ \text{first, second, third} \end{pmatrix}$ kind. Let x_1, x_2, x_3 denote the number of dresses of the first, second, and third kind, respectively.

$$f_1(a) = \max_{0 \le x_3 \le [\frac{a}{5}]} [4 x_3] = 4 [\tfrac{a}{5}]$$

$$f_2(a) = \max_{0 \le x_2 \le [\frac{a}{3}]} [2.5 x_2 + f_1 (a-3x_2)]$$

$$f_3(a) = \max_{0 \le x_1 \le [\frac{a}{2}]} [1.5 x_1 + f_2 (a-2x_1)]$$

where x_1, x_2, x_3 are further restricted to be integers, and $[\frac{a}{k}]$ denotes the largest integer less than or equal to $\frac{a}{k}$.

The possible values of $f_1(a)$, $f_2(a)$ and $f_3(a)$ are tabulated below.

a	$f_1(a)$
0	0
1	0
2	0
3	0
4	0
5	4
6	4
7	4
8	4
9	4
10	8
11	8
12	8
13	8
14	8
15	12
16	12
17	12
18	12
19	12

a \ x_2	0	1	2	3	4	5	6	$f_2(a)$
0	0							0
1	0							0
2	0							0
3	0	2.5						2.5
4	0	2.5						2.5
5	4	2.5						4
6	4	2.5	5					5
7	4	2.5	5					5
8	4	6.5	5					6.5
9	4	6.5	5	7.5				7.5
10	8	6.5	5	7.5				8
11	8	6.5	9	7.5				9
12	8	6.5	9	7.5	10			10
13	8	10.5	9	7.5	10			10.5
14	8	10.5	9	11.5	10			11.5
15	12	10.5	9	11.5	10	12.5		12.5
16	12	10.5	13	11.5	10	12.5		13
17	12	10.5	13	11.5	14	12.5		14
18	12	14.5	13	11.5	14	12.5	15	15
19	12	14.5	13	15.5	14	12.5	15	15.5

a \ x_1	0	1	2	3	4	5	6	7	8	9	$f_3(a)$
19	15.5	15.5	15.5	15	15	15	14	14.5	14.5	13.5	15.5

(a) The maximum profit is

$$f_3(19) = 15.5 \text{ dollars}$$

and is realizable in the following ways.

x_1	x_2	x_3
0	3	2
1	4	1
2	5	0

(b) The largest integer profit is 15 dollars, and is realizable in the following ways.

x_1	x_2	x_3
3	1	2
4	2	1
5	3	0

□

13-10. An electronics firm has a contract to deliver the following number of radios in a 4-month period.

	1st month	2nd month	3rd month	4th month
Number of radios	300	400	300	300

The labor costs are

	1st month	2nd month	3rd month	4th month
Labor cost per radio	\$10	\$10	\$12	\$13

The inventory cost for each month is \$1.5 for each radio in stock at the beginning of the month. The cost of setting up production each month is \$250. The number of radios produced must be an integer multiple of 100. Determine an optimal policy.

Let $f_k(a)$ denote the minimum operating cost for the last k months, with a radios in stock at the beginning of this period.

$$f_1(a) = \begin{cases} 1.5a + 250 + 13\,(300-a) & 0 \le a < 300 \\ 1.5a & a = 300 \end{cases}$$

$$f_2(a) = \min_{\max(0,300-a) \le x_3 \le 600-a} \left[1.5a + \left\langle \begin{matrix} 250 + 12x_3 \\ 0 \end{matrix} \right\rangle + f_1(a + x_3 - 300)\right]$$

$$f_3(a) = \min_{\max(0,\ 400-a) \le x_2 \le 1000-a} \left[1.5a + \left\langle \begin{matrix} 250 + 10x_2 \\ 0 \end{matrix} \right\rangle + f_2\ (a + x_2 - 400)\right]$$

$$f_4(a) = \min_{\max(0,\ 300-a) \le x_1 \le 1300-a} \left[1.5a + \left\langle \begin{matrix} 250 + 10x_1 \\ 0 \end{matrix} \right\rangle + f_3\ (a + x_1 - 300)\right]$$

where

$$\left\langle \begin{matrix} 250 + kx_i \\ 0 \end{matrix} \right\rangle = \begin{cases} 250 + kx_i & \text{if } x_i > 0 \\ 0 & \text{if } x_i = 0 \end{cases}$$

The possible values of $f_1(a)$, $f_2(a)$, $f_3(a)$ and $f_4(a)$ are tabulated below.

a	$f_1(a)$
0	4150
100	3000
200	1850
300	450

a \ x_3	0	100	200	300	400	500	600	$f_2(a)$
0				8000	8050	8100	7900	7900
100			6950	7000	7050	6850		6850
200		5900	5950	6000	5800			5800
300	4600	4900	4950	4750				4600
400	3600	3900	3700					3600
500	2600	2650						2600
600	1350							1350

a \ x_2	0	100	200	300	400	500	600	700	800	900	1000	$f_3(a)$
0					12,150	12,100	12,050	11,850	11,850	11,850	11,600	11,600
100				11,300	11,250	11,200	11,000	11,000	11,000	10,750		10,750
200			10,450	10,400	10,350	10,150	10,150	10,150	9900			9900
300		9600	9550	9500	9300	9300	9300	9050				9050
400	8500	8700	8650	8450	8450	8450	8200					8200
500	7600	7800	7600	7600	7600	7350						7350
600	6700	6750	6750	6750	6500							6500
700	5650	5900	5900	5650								5650
800	4800	5050	4750									4750
900	3950	3950										3950
1000	2850											2850

a \ x_1	0	100	200	300	400	500	600	700	800	900
0				14,850	15,000	15,150	15,300	15,450	15,600	15,750

1000	1100	1200	1300	$f_4(a)$
15,900	16,000	16,200	16,100	14,850

The minimum cost for the four-month operation is

$$f_4(0) = 14{,}850 \text{ dollars}$$

The optimal production schedule is

Month	1	2	3	4
Number of radios	300	1000	0	0

□

13-11. To advertise a new product, a company has the choice of purchasing commercial time from a radio station or from a television station. The subscription is to be made on a weekly basis. Determine an optimal policy for the operation during a three-week period, given the following data:

1. The estimated profit from sales during the dth consecutive week of radio (television) advertisement is 4,000/d (6,000/d) dollars.
2. There is a fee of $2,500 each time the company switches its advertising medium.

Let $f_k(a,0)$ ($f_k(0,a)$) denote the maximum profit realizable during the final k weeks, given that there has been a consecutive weeks of radio (television) advertising immediately preceding this period.

$$f_1(1,0) = \max\left(\frac{4000}{2},\ 6000 - 2500\right) = 3500$$

$$f_1(2,0) = \max\left(\frac{4000}{3},\ 6000 - 2500\right) = 3500$$

$$f_1(0,1) = \max\left(\frac{6000}{2},\ 4000 - 2500\right) = 3000$$

$$f_1(0,2) = \max\left(\frac{6000}{3},\ 4000 - 2500\right) = 2000$$

$$f_2(1,0) = \max\left[\frac{4000}{2} + f_1(2,0),\ 6000 - 2500 + f_1(0,1)\right]$$

$$= \max\left[2000 + 3500,\ 3500 + 3000\right]$$

$$= 6500$$

$$f_2(0,1) = \max\left[\frac{6000}{2} + f_1(0,2),\ 4000 - 2500 + f_1(1,0)\right]$$

$$= \max\left[3000 + 2000,\ 1500 + 3500\right]$$

$$= 5000$$

$$f_3(0,0) = \max [4000 + f_2(1,0),\ 6000 + f_2(0,1)]$$

$$= \max [4000 + 6500,\ \ 6000 + 5000]$$

$$= 11{,}000$$

The maximum profit realizable during a three week period is

$$f_3(0,0) = 11{,}000 \text{ dollars}$$

There are two optimal advertising policies

Week	1	2	3
Medium	Television	Television	Television
	Television	Radio	Television

□

13-12. (a) Consider the operation of a machine shop in which there are 200 machines. A machine can be used to manufacture product A for a profit of $300 per week of product B for a profit of $400 per week. After each week's operation, 30 percent of the machines making product A are inoperable, and 50 percent of the machines making product B are inoperable. Find an optimal policy of operation for a four-week period.

(b) Several mechanics are available to repair all inoperable machines resulting only from the second week's operation during the weekend preceding the third week. When repaired machines are used to manufacture product A, 50 percent of them become inoperable after one week. For product B, 80 percent becomes inoperable after one week. The mechanics can be hired for $36,000. Should they be hired?

(c) Repeat part (b) if the mechanics can repair all inoperable machines from previous weeks.

(a) Let $f_k(a)$ denote the maximum profit from k weeks of operation, with a operable machines available at the beginning of this period.

$$f_1(a) = \max_{0 \le x \le a} [300x + 400\ (a-x)]$$

$$= \max_{0 \le x \le a} [-100x + 400a]$$

$$= 400a \qquad (\text{at } x = 0)$$

$$f_2(a) = \max_{0 \le x \le a} [300x + 400\ (a-x) + f_1\ (0.7x + 0.5\ (a-x))]$$

$$= \max_{0 \le x \le a} [-20x + 600a]$$

$$= 600a \qquad \text{(at } x = 0\text{)}$$

$$f_3(a) = \max_{0 \le x \le a} [300x + 400\ (a-x) + f_2(0.7x + 0.5\ (a-x))]$$

$$= \max_{0 \le x \le a} [20x + 700]$$

$$= 720a \qquad \text{(at } x = a\text{)}$$

$$f_4(a) = \max_{0 \le x \le a} [300x + 400\ (a-x) + f_3(0.7x + 0.5\ (a-x))]$$

$$= \max_{0 \le x \le a} [44x + 760a]$$

$$= 804a \qquad \text{(at } x = a\text{)}$$

The maximum profit from 4 weeks of operation is

$$f_4(200) = 160{,}800 \text{ dollars}$$

The optimal policy is

Week	1	2	3	4
Product A	200	140	0	0
Product B	0	0	98	49

(machines assigned)

(b) Let $g_k(a)$ denote the maximum profit from the last k weeks of operation, with a repaired machines available at the beginning of this period.

$$g_1(a) = \max_{0 \le x \le a} [300x + 400\ (a-x)]$$

$$= \max_{0 \le x \le a} [-100x + 400a]$$

$$= 400a \qquad \text{(at } x = 0\text{)}$$

$$g_2(a) = \max_{0 \le x \le a} [300x + 400\ (a-x) + g_1\ (0.5x + 0.2\ (a-x))]$$

$$= \max_{0 \le x \le a} [20x + 480a]$$

$$= 500a \qquad \text{(at } x = a\text{)}$$

Let $f_1(a)$ and $f_2(a)$ be as formulated in part (a). $f_3(a)$ and $f_4(a)$ are re-formulated as follows:

$$f_3(a) = \max_{0 \le x \le a} [300x + 400\ (a-x) + f_2\ (0.7x + 0.5\ (a-x)) + g_2\ (0.3x + 0.5\ (a-x))]$$

$$= \max_{0 \le x \le a} [-80x + 950a]$$

$$= 950a \qquad (\text{at } x = 0)$$

$$f_4(a) = \max_{0 \le x \le a} [300x + 400\ (a-x) + f_3\ (0.7x + 0.5\ (a-x))]$$

$$= \max_{0 \le x \le a} [90x + 875]$$

$$= 965a \qquad (\text{at } x = a)$$

The maximum profit from 4 weeks of operation is

$$f_4\ (200) - 36{,}000 = 193{,}000 - 36{,}000$$
$$= 157{,}000 \quad \text{dollars}$$

The optimal policy is

	Week	1	2	3	4
not repaired	Product A	200	0	0	0
	Product B	0	140	70	35
repaired	Product A	0	0	70	0
	Product B	0	0	0	35

(machines assigned)

Since the maximal profit is less than that of part (a), the mechanics should not be hired.

(c) Let $f_1(a)$ and $f_2(a)$ be as formulated in part (a).
Let $g_1(a)$ and $g_2(a)$ be as formulated in part (b).

Let $f_3(a,b)$ denote the maximal profit from the last 3 weeks of operation, with a operable machines and b inoperable machines available at the beginning of this period.

$$f_3(a,b) = \max_{0 \le x \le a} [300x + 400\ (a-x) + f_2\ (0.7x + 0.5\ (a-x)) + g_2\ (0.3x + 0.5\ (a-x) + b)]$$

$$= \max_{0 \le x \le a} [-80x + 950a + 500b]$$

$$= 950a + 500b \qquad (\text{at } x = 0)$$

$$f_4(a) = \max_{0 \le x \le a} [300x + 400\,(a-x) + f_3\,(0.7x + 0.5\,(a-x),\ 0.3x + 0.5\,(a-x))]$$

$$= \max_{0 \le x \le a} [-10x + 1125a]$$

$$= 1125a \qquad (\text{at } x = 0)$$

The maximum profit from 4 weeks of operation is

$$f_4(200) - 36{,}000 = 225{,}000 - 36{,}000$$

$$= 189{,}000 \quad \text{dollars}$$

The optimal policy is

	Week	1	2	3	4
not repaired	Product A	0	0	0	0
	Product B	200	100	50	25
repaired	Product A	0	0	150	0
	Product B	0	0	0	75

(machines assigned)

Since the maximal profit is greater than that of part (a), the mechanics should be hired.

□

14-1. Let A be a $(t + 2) \times n^2$ array with the integers 1, 2, $\cdots$, n as entries ($n \geq 3$, and $t \geq 2$). Prove that if the columns of every $2 \times n^2$ sub-array of A are the n^2 ordered pairs of the integers 1, 2, $\cdots$, n, then there is a set of t orthogonal Latin squares of order n corresponding to A.

HINT: The columns of A can be permuted so that the first row will read 11 $\cdots$ 1 22 $\cdots$ 2 $\cdots$ nn $\cdots$ n and the second row will read 12 $\cdots$ n12 $\cdots$ n $\cdots$ 12 $\cdots$ n. Each of the remaining rows can be made to correspond to a Latin square in the orthogonal set.

For each of the remaining t rows of A

$$[a_{j,1}\ a_{j,2} \cdots a_{j,n^2}] \qquad j = 3, 4, \cdots, t + 2$$

form the $n \times n$ array B_j

$$B_j = \begin{bmatrix} a_{j,1} & a_{j,2} & \cdots & a_{j,n} \\ a_{j,n+1} & a_{j,n+2} & \cdots & a_{j,2n} \\ \vdots & \vdots & & \vdots \\ a_{j,n^2-n+1} & a_{j,n^2-n+2} & \cdots & a_{j,n^2} \end{bmatrix}$$

Each B_j, $j = 3, 4, \cdots, t + 2$, is a latin square, since

(i) The sub-array of A consisting of the 1st and jth rows

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 2 & 2 & \cdots & 2 & \cdots & n & n & \cdots & n \\ a_{j,1} & a_{j,2} & \cdots & a_{j,n} & a_{j,n+1} & a_{j,n+2} & \cdots & a_{j,2n} & \cdots & a_{j,n^2-n+1} & a_{j,n^2-n+2} & \cdots & a_{j,n^2} \end{bmatrix}$$

insures that no two elements in the same row of B_j are identical.

(ii) The sub-array of A consisting of the 2nd and jth rows

$$\begin{bmatrix} 1 & 2 & \cdots & n & 1 & 2 & \cdots & n & \cdots & 1 & 2 & \cdots & n \\ a_{j,1} & a_{j,2} & \cdots & a_{j,n} & a_{j,n+1} & a_{j,n+2} & \cdots & a_{j,2n} & \cdots & a_{j,n^2-n+1} & a_{j,n^2-n+2} & \cdots & a_{j,n^2} \end{bmatrix}$$

insures that no two elements in the same column of B_j are identical.

The set of latin squares $\{B_3, B_4, \cdots, B_{t+2}\}$ is an orthogonal set, since for any two of these latin squares B_i and B_j, the sub-array of A consisting of the ith and jth rows insures that B_i and B_j are orthogonal.

□

14-2. Show that there exists no (b,v,r,k,λ)-configuration with the following parameters:

(a) $b = 8,\ v = 6,\ r = 5,\ k = 3,\ \lambda = 2$

(b) $b = 22,\ v = 22,\ r = 7,\ k = 7,\ \lambda = 2$

(a) The necessary condition

$$bk = vr$$

of Theorem 14-5 is violated.

(b) $b = v$ and $r = k$, so that this would be a (v,k,λ)-configuration.

However, v is even and $k - \lambda$ is not the square of an integer, violating the necessary condition of Theorem 14-8.

□

14-3. (a) A balanced incomplete block design has the parameters $v = 15$, $k = 5$, and $\lambda = 2$. Determine b and r.

(b) A balanced incomplete block design has the parameters $v = 21$, $b = 28$, and $r = 8$. Determine k and λ.

Using the two relations

$$bk = vr$$
$$r(k-1) = \lambda(v-1)$$

of Theorem 14-5:

(a) $$r = \frac{\lambda(v-1)}{k-1} = \frac{2 \cdot 14}{4} = 7$$

$$b = \frac{vr}{k} = \frac{15 \cdot 7}{5} = 21$$

(b) $$k = \frac{vr}{b} = \frac{21 \cdot 8}{28} = 6$$

$$\lambda = \frac{r(k-1)}{v-1} = \frac{8 \cdot 5}{20} = 2$$

□

14-4. Four of the seven blocks of a $(7,3,1)$-configuration are $\{1,2,4\}$, $\{2,3,5\}$, $\{3,4,6\}$, and $\{4,5,7\}$. Find the remaining blocks.

$\{1,5,6\}$, $\{2,6,7\}$, $\{1,3,7\}$

□

14-5. Prove that in a (v,k,λ)-configuration, if v is even, then $k - \lambda$ is the square of an integer. (This is the first part of Theorem 14-8.)

Let Q be the incidence matrix for the (v,k,λ)-configuration.

As shown in the proof of Theorem 14-7, with $r = k$

$$\det QQ^T = [k + (v-1)\lambda](k-\lambda)^{v-1}$$

and, using Theorem 14-5, with $r = k$

$$k(k-1) = \lambda(v-1)$$

Substituting for $\lambda(v-1)$

$$\det QQ^T = k^2 (k-\lambda)^{v-1}$$

but

$$\det QQ^T = \det Q \cdot \det Q^T = (\det Q)^2$$

so

$$(\det Q)^2 = k^2 (k-\lambda)^{v-1}$$

$$\det Q = k (k-\lambda)^{\frac{v-1}{2}}$$

Since $(v-1)$ is odd, and $\det Q$ must be an integer, $(k-\lambda)$ must be the square of an integer.

□

14-6. A Steiner triple system of order v is a (b,v,r,k,λ)-configuration with $k = 3$ and $\lambda = 1$.

(a) Show that $r = (v-1)/2$ and $b = v(v-1)/6$.

(b) Show that $v \equiv 1 \pmod 6$ or $v \equiv 3 \pmod 6$.

(a) From Theorem 4-5

$$r(k-1) = \lambda(v-1) \qquad bk = vr$$

With $k = 3$ and $\lambda = 1$, these equations give

$$2r = v-1 \qquad 3b = vr$$

$$r = \frac{v-1}{2} \qquad b = \frac{v}{3} r = \frac{v(v-1)}{6}$$

(b) Since r and b must be integers, $2|(v-1)$ and $6|v(v-1)$.

$2|(v-1)$ implies that $v \equiv 1$ or 3 or 5 (mod 6).

$2|(v-1)$ and $6|v(v-1)$ implies that $3|v(\frac{v-1}{2})$, which implies that $3|v$ or $3|(\frac{v-1}{2})$, which implies that $v \equiv 0$ or 1 or 3 (mod 6).

Thus, $v \equiv 1$ or 3 (mod 6).

□

14-7. Let S_1 be a Steiner triple system of order v_1, and let S_2 be a Steiner triple system of order v_2. Let $\{a_1, a_2, \cdots, a_{v_1}\}$ be the set of objects in S_1, and let $\{b_1, b_2, \cdots, b_{v_2}\}$ be the set of objects in S_2. Let X be a set containing $v_1 v_2$ objects c_{ij} ($i = 1,2,\cdots, v_1$; $j = 1,2,\cdots, v_2$). Let S be a set of 3-subsets of X, $\{c_{pq}, c_{tu}, c_{yz}\}$, each satisfying one of the following conditions:

1. $q = u = z$, and $\{a_p, a_t, a_y\}$ is in S_1.
2. $p = t = y$, and $\{b_q, b_u, b_z\}$ is in S_2.
3. $\{a_p, a_t, a_y\}$ is in S_1, and $\{b_q, b_u, b_z\}$ is in S_2.

Show that S is a Steiner triple system of order $v_1 v_2$.

First show that each element in X appears in the same number (r) or blocks in S.

Let c_{pq} be any element in X.

Any block in S must satisfy exactly one of the conditions (1), (2) or (3) of the problem statement. Among the blocks in S containing c_{pq}, there are:

r_1 satisfying condition (1)

r_2 satisfying condition (2)

$2r_1r_2$ satisfying condition (3)

(The last statement follows from the observation that $\{a_p, a_t, a_y\} \in S_1$ and $\{b_q, b_u, b_z\} \in S_2$ produce both $\{c_{pq}, c_{tu}, c_{yz}\}$ and $\{c_{pq}, c_{tz}, c_{yu}\}$ in S)

So, any element $c_{pq} \in X$ appears in $r_1 + r_2 + 2r_1r_2$ blocks in S, that is $r = r_1 + r_2 + 2r_1r_2$.

Next show that any pair of distinct elements in X appear together in the same number (λ) of blocks in S.

Let c_{pq} and c_{tu} be any two distinct elements in X.

If $p \neq t$, $q = u$ then condition (1) must hold. Since there are λ_1 blocks in S_1 containing both a_p and a_t, there are λ_1 blocks in S containing both c_{pq} and c_{tu}.

If $p = t$, $q \neq u$ then condition (2) must hold. Since there are λ_2 blocks in S_2 containing both b_q and b_u, there are λ_2 blocks in S containing both c_{pq} and c_{tu}.

If $p \neq t$, $q \neq u$ then condition (3) must hold. There are $\lambda_1\lambda_2$ ordered pairs of blocks, one from each of S_1 and S_2, the first containing both a_p and a_t, the second containing both b_q and b_u. Thus, there are $\lambda_1\lambda_2$ blocks in S containing both c_{pq} and c_{tu}.

So, any pair of distinct elements in X appear together in λ_1 or λ_2 or $\lambda_1\lambda_2$ blocks in S. But since $\lambda_1 = \lambda_2 = 1$, any such pair appear together in exactly one block in S, that is $\lambda = 1$

Finally, it is clear that $k = 3$.

Thus, S is a (b,v,r,3,1)-configuration, where

$$v = v_1 v_2$$

$$r = r_1 + r_2 + 2r_1 r_2$$

(b has not been computed). Therefore, S is a Steiner triple system of order $v_1 v_2$.

□

Professor Liu's book is an undergraduate textbook, the outgrowth of a set of class notes which he used in teaching a course in applied combinatorial mathematics in the Electrical Engineering Department at the Massachusetts Institute of Technology.

Combinatorial mathematics is useful in industry, particularly in communication engineering, operational research, statistics and computing. Although a sizeable fraction of the output of mathematical departments goes into industrial mathematics, there has been relatively little concentration upon c ombinatorial mathematics in the undergraduate syllabus. There are two main reasons for this. First, the subject is not very well suited to the traditional type of mathematics lecture, composed of an orderly and didactically satisfying sequence of definitions and theorems. Second, it is " hard " as opposed to " soft" mathematics: the problems are readily understood, but they call for a fairly high level of manipulative skill; they are too much like "scholarship questions" and they can easily be too difficult for the run-of-the-mill candidate in a Final Honours examination.

But it is a pity that mathematicians should opt out of difficulties, and that they should shy away from useful mathematics. So Professor Liu's book, which is distinguished with a wealth of examples and problems in a wide range of fields, should do much to redress the balance. For the weaker brethren, there is a crib by Mr. Edelberg to the solution of these problems; though naturally students should refer to it only after they have made a serious attempt on their own to grapple with the problems.

Briefly, the contents of Professor Liu's book are:

Chapter 1. A straightforward account of permutations and combinations. Stirling's formula, obtained by means of inequalities and the remark that the derivation of the constant $\sqrt{(2\pi)}$ can be "looked up in a textbook on advanced calculus".

Chapter 2. Generating functions, with particular reference to combinations, permutations, and partitions. Ferrers graphs.

Chapter 3. Recurrence relations. Solution of simple difference equations, including some non-linear equations arising from the multiplication of generating functions.

Chapter 4. Principle of inclusion and exclusion. Permutations under various restrictions. Generating polynomials for arrangements of "mutually non-capturing" rooks on a chessboard: the treatment of these rook polynomials bears some resemblance to (and might be used to introduce) generating functions in statistical mechanics.

Chapter 5. Polya's theory of counting. Treatment of permutation groups leading eventually to de Bruijn's generalization of Polya's theorem. This chapter is considerably more difficult than the rest of the book and will probably prove tough going for the average undergraduate.

Chapter 6. Fundamental concepts in the theory of graphs. Incidence relations, Euler and Hamiltonian paths.

Chapter 7. Trees, circuits and cut-sets. (A comparatively dull chapter.)

Chapter 8. Planar and dual graphs. Euler's formula. Kuratowski's theorem.

Chapter 9. Domination, independence and chromatic numbers. Application to transmission of messages in noisy channels.

Chapter 10. Transport networks. Max-flow/Min-cut theorem. Feasible flows.

Chapter 11. Matching theory. Bipartite graphs. Maximal matching.

Chapter 12. Linear programming. (Rather traditional approach.)

Chapter 13. Dynamic programming. (Necessarily sketchy, but quite good as an introduction at the undergraduate level.)

Chapter 14. Block designs. Orthogonal Latin squares. Balanced incomplete graphs.

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